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Exact *S*-matrices for supersymmetric sigma models and the Potts model

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Abstract
We study the algebraic formulation of exact factorizable *S*-matrices for integrable two-dimensional field theories. We show that different formulations of the *S*-matrices for the Potts field theory are essentially equivalent, in the sense that they can be expressed in the same way as elements of the Temperley–Lieb algebra, in various representations. This enables us to construct the *S*-matrices for certain nonlinear sigma models that are invariant under the Lie 'supersymmetry' algebras $\mathrm{sl}(m+n|n)$ $(m = 1, 2, n > 0)$, both for the bulk and for the boundary, simply by using another representation of the same algebra. These *S*-matrices represent the perturbation of the conformal theory at $\theta = \pi$ by a small change in the topological angle θ. The $m = 1, n = 1$ theory has applications to the spin quantum Hall transition in disordered fermion systems. We also find *S*-matrices describing the flow from weak to strong coupling, both for $\theta = 0$ and $\theta = \pi$, in certain other supersymmetric sigma models.

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1. Introduction

In this paper we consider various models in statistical mechanics and quantum field theory in two dimensions, some well-known and some less familiar. These include the Potts model, restricted solid-on-solid (RSOS) models, quantum spin chains and nonlinear sigma models (the latter two having Lie superalgebras as symmetries). We focus here on the *S*-matrices of integrable quantum field theories. The exact *S*-matrix for the massive particles in an integrable quantum field theory can be found by using a set of well-known criteria [1]. For example, the multi-particle *S*-matrix elements factorize into sums of products of two-particle elements. The consistency criterion for the factorization is called the Yang–Baxter equation.

The main purpose of this paper is to show that such *S*-matrices can be constructed, and the Yang–Baxter equation verified, by working with elements of certain abstract algebras. Like

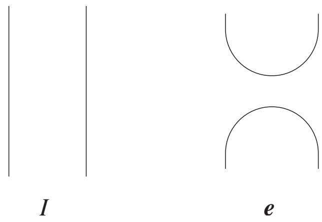

Figure 1. The generators of the TL algebra.

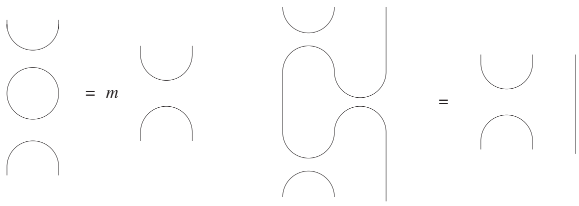

Figure 2. The relations in the TL algebra.

any element of the algebra, the S-matrix can be expressed in terms of the generators. The S-matrices of various theories are images of the same abstract S-matrix, in the representation appropriate to each theory, under the homomorphism of the abstract algebra into the algebra of linear maps on the vector space of internal states of the particles in each theory. These S-matrices can conveniently be written in terms of the generators of (the image of) the algebra in each case. This allows us to use the 'same' S-matrices to solve different problems. Also, because one can often show that the partition function of the model follows solely from the algebraic properties of the S-matrix, there can be different representations giving the same thermodynamics. This idea is very similar to that of Temperley and Lieb, who studied lattice models algebraically within the transfer matrix formulation, and introduced a different representation of their algebra in order to find the partition function [2].

The algebras that arise in the cases we study here are the Temperley–Lieb (TL) [2] and Murakami–Birman–Wenzl (BMW) [3] algebras. In fact, following old work in lattice models [4, 5], we construct the SO(3) version of the latter within the former. As the TL algebra is of central importance, we introduce it now. Its generators are labelled e_i, with $i = 0, \ldots, 2N-2$. They obey the relations [2, 6]

$$e_i^2 = me_i \qquad e_ie_{i\pm1}e_i = e_i \qquad e_ie_j = e_je_i \quad (j \neq i, i \pm 1). \tag{1}$$

The relations (1) can be viewed as those arising from non-intersecting lines: the identity I and the generators e_i acting on the two sites $i, i+1$ can be represented as in figure 1, while the algebraic relations are shown in figure 2. This non-intersection will be crucial in what follows. (We have defined the TL algebra with an odd number of generators, as this will be what we usually need, but the analogue with an even number of generators exists.)

The algebraic approach was already utilized in [7] to study S-matrices proposed to describe O(m)-invariant field theories for $|m| \leqslant 2$ [8]. The O(m)S-matrices were, strictly speaking, valid only for $m = 1$ and 2, although they were cleverly written in a form which could seemingly be applicable to continuous m. It was shown in [7] how to formulate these S-matrices in terms of the TL algebra. Using a different representation of this algebra gives the S-matrices of the sine-Gordon model. This enabled the thermodynamics of the O(m) model in its high-temperature phase to be derived for any $|m| < 2$ [9]. This includes the interesting limit $m \to 0$, where the model describes polymers (self-avoiding random walks). In addition, yet another representation for the TL algebra yielded the S-matrices for the ϕ_{13} perturbations of the minimal models of conformal field theory [10].

One of the key advantages of using the algebraic formulation of the S-matrices (or that for the partition function of a lattice model) is that it allows one to precisely define and calculate in theories such as this '$m \to 0$' limit. One of our main motivations for this work was to be able to study the problem of percolation. Percolation is often defined as the $Q \to 1$ limit of the Q-state Potts model. This limit can be defined precisely in the lattice model when the TL (also known as the six-vertex, or XXZ chain) representation is used (see section 2.1), and in this representation the continuum field theory at the critical point can be described by a Coulomb gas [11–13]. One can define the continuum field theory off the critical point via perturbation theory, but the algebraic formulation of the S-matrix is required to solve the theory.

In order to describe the percolation limit $Q \to 1$, one needs an S-matrix valid for all Q. Two seemingly different algebraic formulations of the Potts S-matrix have been proposed in [14, 15]. The first was given in [14], where such an S-matrix was found by studying the quantum affine algebra $U_q\big(A_2^{(2)}\big)$. At special values of Q, this S-matrix describes the integrable Φ_{21} perturbation of the minimal models of conformal field theory [16], described in section 2.2. This formulation is precise, but this S-matrix is quite intricate, and its intuitive connection to the original Potts model is not very clear. An elegant S-matrix for the Potts model in the continuum limit was proposed in [15]. Here the connection to the original Potts representation of the lattice model is quite clear: the world lines of the particles represent domain walls between regions of different Potts spin states. Unfortunately, this version of this S-matrix for arbitrary Q is not precise: the algebra is not written out explicitly, and the representations are defined only for Q an integer. Moreover its connection to the original S-matrix of [14] is not clear.

In this paper, we show how the S-matrices of [14, 15] can be written in terms of the same algebra, with elements in different representations. We then use this algebraic formulation to find the S-matrix for another set of field theories. These are nonlinear sigma models, where the fields take values on the (super-) manifolds $\mathbf{CP}^{m-1} \cong \mathrm{U}(m)/\mathrm{U}(1)\times \mathrm{U}(m-1)$, or $\mathbf{CP}^{m+n-1|n} \cong \mathrm{U}(m+n|n)/\mathrm{U}(1)\times \mathrm{U}(m+n-1|n)$ in a version with $\mathrm{sl}(m+n|n)$ Lie superalgebra symmetry ('supersymmetry'), as described in section 2.3. The actions of these models contain a coupling constant g_σ and θ, the coefficient of the topological (instanton) term. For $0 < m \leqslant 2$, there is a critical point at $\theta = \pi \pmod{2\pi}$, and the critical properties are independent of n for each m. If one perturbs the critical theory by moving θ away from π, one obtains a massive field theory. In this paper, we find the exact S-matrices for these field theories when $m = 1$, which corresponds to the percolation case, and $m = 2$, for general $n > 0$.

A particular application of these results occurs in a disordered noninteracting fermion model in two dimensions, in which there is a spin quantum Hall transition [17–21], similar to but distinct from the better-known integer quantum Hall transition. The problem can be mapped to a nonlinear sigma model on the target manifold $\mathrm{OSp}(2n|2n)/\mathrm{U}(n|n)$ [22], which is $\cong \mathbf{CP}^{1|1}$ for $n = 1$. Indeed, a lattice version of the problem [17] maps directly onto the corresponding supersymmetric vertex model, and hence to percolation [18]. Our results give the exact solution (that is, the S-matrices), both for the bulk and the boundary, of the spin quantum Hall transition in the scaling limit close to but *off* criticality.

We begin in section 2 by reviewing the models to be analysed. In section 3, we show that the two earlier formulations of the S-matrix of the Potts model, though written in different representations, are algebraically the same. We do this by writing each S-matrix in terms of generators of the SO(3) BMW algebra. In section 4, we show that this algebra can be represented within the TL algebra, thus giving another way of describing the S-matrix.

This makes it possible to write down the S-matrices in the language of the sigma models. These results also allow us in section 5 to characterize the boundary S-matrix for free boundary conditions, originally given in [23], and extend this also to the sigma models. This gives an exact description of properties involving the edge states in the spin quantum Hall effect. In section 6, we show that other S-matrices that can be written in terms of the TL algebra can also be applied to certain supersymmetric sigma models, which hence are integrable on the lines $\theta = 0$ and $\theta = \pi$ as well as for the perturbation of θ away from π, but perhaps do not apply to the $\mathbf{CP}^{m+n-1|n}$ sigma models. In the appendix, we use our earlier results to study S-matrices invariant under SU(m) for $m > 2$. In particular, this allows us to find a unitary, crossing-symmetric S-matrix for particles in the adjoint representation of SU(m), which however has some unphysical properties that prevent it from representing the corresponding sigma models.

2. The models and their phenomenology

In this section we introduce and review the various models to be discussed in this paper.

2.1. The Potts model

We begin with the Q-state Potts model, well-known in two-dimensional classical statistical mechanics. It is defined by placing a 'spin' s_i that takes values $1, 2, \ldots, Q$ ($Q \geqslant 0$ a integer) at every site (or node) $i \in \mathbf{Z}^2$ of the infinite square lattice (or graph), which we also call $\mathbf{Z}^2$. The Hamiltonian is a sum of nearest neighbour terms only, and is constructed to be invariant under the permutation (or symmetric) group S_Q. This forces it to take the form

$$E_{\text{Potts}} = -\sum_{\langle ij\rangle} \delta_{s_i s_j}$$

up to possible multiplicative and additive constants. The $Q = 2$ case is equivalent to the Ising model. The Hamiltonian can be extended to include 'magnetic field' or source terms for the spin variables, by adding the term $\sum_{i,s_i} h_{s_i}$, with h_{s_i} a set of parameters (the sum $\sum_{s_i} h_{s_i}$ is redundant for each i and can be set to zero). Then correlation functions of arbitrary functions of any set of spins s_i can be calculated using derivatives of $\ln Z$ with respect to the h_{s_i}'s, where

$$Z = \sum_{\{s_i=1,\ldots,Q:i\in\Lambda\}} \mathrm{e}^{-\beta E_{\text{Potts}}}$$

and the sum is over all configurations of the spins, and the sites i are restricted to a finite connected subset Λ of $\mathbf{Z}^2$.

The partition function (with all $h_{s_i} = 0$) can be rewritten as

$$Z = \sum_{\{s_i=1,\ldots,Q:i=1,\ldots\}} \prod_{\langle ij\rangle} \left(1 + \delta_{s_i s_j}(\mathrm{e}^{\beta} - 1)\right).$$

The product can be multiplied out, and each term can be represented as a (possibly disconnected) graph on the lattice by drawing a line between sites i and j if $\delta_{s_i s_j}$ is present in this term. For a given graph, the sum over spins can now be easily done: it just results in a factor Q^{n_c}, where n_c is the number of connected components (clusters) in the graph. The partition function is now

$$Z = \sum_{\text{graphs}} Q^{n_c}(\mathrm{e}^{\beta} - 1)^{n_l} \tag{2}$$

where n_l is the number of links on the graph. In this form, Q appears only as a parameter, and could be any complex number. We take equation (2) to be the definition of the Q-state

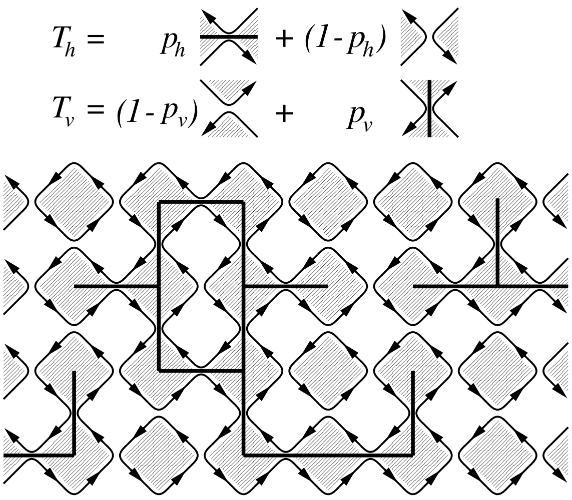

Figure 3. Relation of transfer matrix, graphical expansion of the Potts model and loops on the medial graph.

Potts model for all Q. Strictly speaking, we have defined a different statistical problem, in which the configurations are graphs on the subset Λ of the square lattice, and the weight of each configuration is $Q^{n_c}(\mathrm{e}^{\beta}-1)^{n_l}$. The partition function Z is a starting point for obtaining topological or geometric properties of the graphs. For example, when $Q = 1$ and $\beta \geqslant 0$, this describes bond percolation (the probability of a bond being occupied being $1 - \mathrm{e}^{-\beta}$). When Q is a natural number, the graph formulation is equivalent to the original Potts model, as far as thermodynamic (including the simpler of the magnetic) properties—those that can be obtained by differentiating $\ln Z$—are concerned. However, in general it would be necessary to decorate the graphs if information about the general Potts spin correlations as defined above were required. These distinctions between formulations, though they may appear unimportant, will be very relevant to our discussion, as we will see. In the following, when we say that different models (or different representations for the same transfer matrix or S-matrices) are thermodynamically the same, we will mean the extensive part of the scaling limit of $\ln Z$, in the absence of the symmetry-breaking h_{s_i} source terms, or their analogues in other models.

The Potts model possesses a duality that exchanges large and small β; the ferromagnetic (i.e. $\beta > 0$) Potts model in the thermodynamic limit, in which the square lattice becomes essentially all of $\mathbf{Z}^2$ (with all $h_{s_i} = 0$ from here on), has a phase transition at its self-dual point $\beta = \beta_c$, $\mathrm{e}^{\beta_c} - 1 = \sqrt{Q} \geqslant 0$, $Q \geqslant 0$. The model has long been known to be solvable at this transition. As we will briefly review, it can be mapped onto the solvable six-vertex model by using algebraic techniques [2, 6] based on the transfer matrix approach, and as a result, a number of interesting quantities can be computed exactly [6]. One starting point for this mapping is the graph formulation, equation (2). The set of graphs on the square lattice is in one-to-one correspondence with a different graphical problem, of configurations of non-intersecting loops that fill the links of another square lattice (see figure 3; ignore the arrows on the loops for now). The loops are allowed to touch at each node of the lattice, so there are two ways they can do so at each node. This square lattice is the medial graph of the original one; it has a node at the midpoint of each link of the original lattice, joined by a link to another such node only if they are on links of the original lattice that border a common face [6]. The two possible configurations of lines at each node of the medial graph determine the entire loop configuration. If the loops at a node of the medial graph cut the link of the original lattice, there is no line on that link in the original graph, while if they do not, there is. This establishes the bijection between graphs on the original square lattice, and non-intersecting loops filling the medial square lattice [6].

In the transfer matrix approach, the partition function is defined (up to a numerical factor) using either a trace or an expectation, depending on the desired boundary conditions, of a power of a transfer matrix, which represents the addition of a row to the lattice. The transfer matrix for the Potts model can be written as

$$T \equiv T_1 T_3 \cdots T_{2N-3} T_0 T_2 \cdots T_{2N-2} \tag{3}$$

for the case of N sites in a row of the Potts model (or $2N$ sites of the model on the medial graph), and free boundary conditions. The operators acting on medial graph sites $i, i+1$ are

$$\begin{aligned} T_i &= p_{\mathrm{v}} + (1-p_{\mathrm{v}})e_i \qquad && (i \text{ even}) \\ T_i &= (1-p_{\mathrm{h}}) + p_{\mathrm{h}} e_i \qquad && (i \text{ odd}) \end{aligned} \tag{4}$$

in terms of the TL generators e_i obeying (1) with $Q = m^2$. (For the ferromagnetic Potts model one has $m > 0$, but in some formulations we can make sense of both signs of m.) In the original Potts model, the operators e_{2i}, e_{2i+1} act on the Potts spin variables $a_i = 1, \ldots, Q$, for Potts sites $i = 0, \ldots, N-1$. The e_{2i}'s act on the single site i, while the e_{2i+1}'s act on the pair $i, i+1$ with both $i, i+1$ in the range $0, \ldots, N-1$ (and on all other sites as the identity in both cases). Explicitly, the matrix elements in this representation, with a_i' labelling rows, a_i labelling columns, are

$$\begin{aligned} (e_{2i})_{a_i', a_i} &= 1/m \\ (e_{2i+1})_{a_i' a_{i+1}', a_i a_{i+1}} &= m \delta_{a_i a_{i+1}} \delta_{a_i' a_i} \delta_{a_{i+1}' a_{i+1}}. \end{aligned} \tag{5}$$

(The e_{2i} here is the $Q \times Q$ matrix with all entries equal to $1/m$.) In the isotropic case, p_{v} and p_{h} are equal, with

$$p_{\mathrm{v}} = p_{\mathrm{h}} = \frac{\mathrm{e}^{\beta} - 1}{\mathrm{e}^{\beta} - 1 + m}.$$

In the percolation case $Q = 1$, p_{v} and p_{h} are the probabilities for the occupation of bonds in, say, the vertical and horizontal directions, respectively. In general, the transition occurs at the self-dual point $p_{\mathrm{v}} = 1 - p_{\mathrm{h}}$, which gives $\mathrm{e}^{\beta} - 1 = m$ in the isotropic case. In figure 3, we illustrate how the relation of the graphical formulation and the nonintersecting loops formulations look in terms of the TL generators.

Temperley and Lieb [2] realized that, because thermodynamic properties (the partition function) can be calculated using only the algebraic relations (1), any representation of the TL algebra can be used. They found one in which each medial graph site i carries a two-dimensional vector space, whereas the Potts representation above has a Q-dimensional space for each *Potts* site. Thus the TL representation can be used for arbitrary (even complex) Q; the resulting transfer matrix is that of the six-vertex model.

In the anisotropic limit $p_{\mathrm{h}} \to 0$ with $(1 - p_{\mathrm{v}})/p_{\mathrm{h}}$ fixed, we can write $T \simeq \mathrm{e}^{-p_{\mathrm{h}}^{1/2}(1-p_{\mathrm{v}})^{1/2} H}$, where the Hamiltonian H is

$$H = \epsilon \sum_{i \text{ even}} (1 - e_i) + \epsilon^{-1} \sum_{i \text{ odd}} (1 - e_i) \tag{6}$$

and $\epsilon = \sqrt{(1 - p_{\mathrm{v}})/p_{\mathrm{h}}}$. Using the TL representation, this is the Hamiltonian of the XXZ chain (here with free boundary conditions) [6]. The Hamiltonian chain and the transfer matrix T can be solved by Bethe ansatz methods, but only at the transition (self-dual) point, which in the Hamiltonian chain becomes $\epsilon = 1$ [6]; for $\epsilon \neq 1$, one says that the nearest-neighbour coupling is staggered. Similarly, in the six-vertex-model language, the Boltzmann weights are staggered in space when $p_\mathrm{v} \neq 1 - p_\mathrm{h}$.

One finds that this XXZ chain with $\epsilon = 1$, or the six-vertex model with $p_\mathrm{v} = 1 - p_\mathrm{h}$, is critical when $0 < Q \leqslant 4$, and non-critical otherwise. Thus for $Q > 4$, the transition through $\epsilon = 1$ (or $p_\mathrm{v} = 1 - p_\mathrm{h}$) in the Potts model is first order, while for $0 < Q \leqslant 4$, it is second order [6].

2.2. The minimal models and the RSOS models

For $Q \leqslant 4$, the Potts model near its critical point can be described in the continuum by a massive field theory. Away from the critical point, the Potts model maps onto a six-vertex model where the Boltzmann weights are staggered from site to site. Such a lattice model is not integrable, so none of the exact computations in, e.g. [6] are applicable here. However, a lattice model not being integrable does not mean that the corresponding field theory is not integrable. A famous example is when a magnetic field is added to the Ising model at the critical temperature. The field theory that describes the continuum limit of this lattice model is integrable even though the lattice model is not [24]. The same is true for the Potts model off the critical point. We will describe this result in this section, but we need to first introduce the restricted solid-on-solid (RSOS) models [25].

The RSOS models are another set of well-known lattice models whose transfer matrix at their critical points can be formulated in terms of the TL algebra. We will discuss in section 3 how the RSOS representations of the TL algebra can be used to construct S-matrices as well as lattice models. In this subsection we will discuss only the lattice models. RSOS models are defined by placing an integer variable called a height on the sites of the lattice. The height variable is restricted to take values from $1, \ldots, p$, and on adjacent sites it is required to obey certain rules. For example, in the original Andrews–Baxter–Forrester (ABF) models [25], heights on adjacent sites must differ by ± 1. The conformal field theories describing the continuum limit of the critical points arising in the ABF models are well understood. These theories are known as the minimal unitary models, and they have central charge (the coefficient of the conformal anomaly) $c = 1 - 6/(p(p+1))$ [16]. More precisely, they are the A series of modular-invariant conformal field theories, which also arise as multicritical Ising models [26]. The $p = 2$ case is trivial, while the $p = 3$ case is the Ising model, the $p = 4$ case is the tricritical Ising model and the $p = 5$ case is the tetracritical Ising model. The limit $p \to \infty$ which gives the $\mathrm{SU}(2)_1$ WZW theory. These identifications require careful attention on the role of periodic boundary conditions when the theories are formulated on a torus, or on a study of the full set of local operators in each theory.

Because their transfer matrices are identical when expressed in terms of the TL generators, the critical point arising in an ABF model is thermodynamically equivalent to that arising in the Potts model when

$$Q = 4\cos^2\left(\frac{\pi}{p+1}\right). \tag{7}$$

When Q given by this formula with p integer is itself an integer (i.e., when $p = 3, 5$, or $p \to \infty$), we can compare the ABF models directly with the Potts model. For $Q = 3, 4$, though the thermodynamics is the same, the critical theories are not identical, but are closely related (the three-state Potts model yields one of the D series of modular-invariant conformal field theories, while the four-state Potts model is a $c = 1$ orbifold theory).

For the TL (six-vertex, or XXZ) representation of the transfer matrix, the corresponding critical field theory can be formulated as a Coulomb gas [11, 12]. In the language of the conformal field theory, this is a free boson with a charge at infinity [13]. For generic Q, the conformal field theory is not unitary, but when Q takes the values in (7) for p an integer $\geqslant 3$, it is possible to truncate or restrict (more properly, project to a quotient space) the model and obtain a nontrivial conformal field theory that is unitary. The restriction can be done either in the conformal field theory, or in the lattice model, and that is essentially what the ABF representation of the TL algebra does—hence the name restricted solid-on-solid model, as the six-vertex model is sometimes known as the solid-on-solid model.

The continuum formulation involving conformal field theory is very useful, because it allows us to define a massive field theory describing the continuum limit of the Potts model near but not at the critical point for any Q. Namely, one considers perturbing the conformal field theory away from the critical point by adding a relevant operator to the action. This is defined at least to all orders in perturbation theory. The perturbing 'energy' operator corresponding to the Potts model is known as Φ_{21} in the usual conventions of the conformal theory [13, 16] (although in [13] the subscripts are backwards from these now-accepted conventions). This definition is useful because it allows one to show that this massive field theory is integrable [24], even though the underlying Potts lattice model is not. This integrable field theory has a factorized S-matrix that obeys the YB equation, which we will discuss in detail in section 3. These considerations apply in any representation of the TL algebra in the underlying lattice model.

2.3. Spin chains and sigma models

Other models related to the TL algebra include SU(m)-invariant spin chains and vertex models. We take a chain of $2N$ sites $i = 0, \ldots, 2N-1$, with an m-dimensional vector space at each site. We consider the even sites to transform as the fundamental (defining) representation of SU(m), and the odd as the dual (here the same as the conjugate) of the fundamental; this is the so-called $m, \overline{m}$ model. We then take the nearest-neighbour interaction to be the unique (up to additive and multiplicative constants) coupling that is invariant under this action of SU(m). It is essentially the invariant bilinear form in the generators of SU(m), one for each of the two sites, up to similar constants again, and is the usual 'Heisenberg coupling' of magnetism. Since the tensor product of m and $\overline{m}$ representations decomposes into the singlet and the adjoint of SU(m), one can choose the constants such that the nearest-neighbour interaction e_i is m times the projection operator onto the singlet for sites $i, i+1$. Then the e_i's satisfy the TL algebra relations (1) [27, 28].

The SU(m) symmetry of the spin chains can be generalized to the Lie superalgebra symmetry sl($m+n|m$) (supersymmetry for short). The states of the supersymmetric chain live in a graded vector space V with $m+n \geqslant 0$ even and $n \geqslant 0$ odd dimensions at each site with i even, and its dual V^* at each site with i odd (here the dual is *not* the same as the conjugate representation) [29]. We will give the details, as this is the basis for many of our later results. The space of states can be represented using boson and fermion oscillators, with constraints. For i even we have boson operators $b_i^a, b_{ia}^\dagger$ such that $\left[b_i^a, b_{jb}^\dagger\right] = \delta_{ij}\delta_b^a$ $(a, b = 1, \ldots, n+m)$, and fermion operators f_i^α, $f_{i\alpha}^\dagger$ such that $\left\{f_i^\alpha, f_{j\beta}^\dagger\right\} = \delta_{ij}\delta_\beta^\alpha$ $(\alpha, \beta = 1, \ldots, n)$. For i odd, we have similarly boson operators $\overline{b}_{ia}, \overline{b}_i^{a\dagger}$ with $\left[\overline{b}_{ia}, \overline{b}_j^{b\dagger}\right] = \delta_{ij}\delta_a^b$ $(a, b = 1, \ldots, n+m)$, and fermion operators $\overline{f}_{i\alpha}, \overline{f}_i^{\alpha\dagger}$ with $\left\{\overline{f}_{i\alpha}, \overline{f}_j^{\beta\dagger}\right\} = -\delta_{ij}\delta_\alpha^\beta$ $(\alpha, \beta = 1, \ldots, n)$. Note the minus sign in the last anticommutator; since our convention is that the † stands for the adjoint, this minus sign implies that the norms of any two states that are mapped onto each other by the action of a single $\overline{f}_{i\alpha}$ or $\overline{f}_i^{\alpha\dagger}$ have opposite signs, and the 'Hilbert' space has an indefinite inner product (pedantically, it is the norm-squared that is negative). The generators e_i are Hermitian with respect to the indefinite inner product. The special case $n = 0$ is the construction in [27, 28] with SU(m) symmetry.

The supersymmetry generators are the bilinear forms $b^{\dagger}_{ia}b^{b}_{i}$, $f^{\dagger}_{i\alpha}f^{\beta}_{i}$, $b^{\dagger}_{ia}f^{\beta}_{i}$, $f^{\dagger}_{i\alpha}b^{b}_{i}$ for i even, and correspondingly $-\overline{b}^{b\dagger}_{i}\overline{b}_{ia}$, $\overline{f}^{\beta\dagger}_{i}\overline{f}_{i\alpha}$, $\overline{f}^{\beta\dagger}_{i}\overline{b}_{ia}$, $\overline{b}^{b\dagger}_{i}\overline{f}_{i\alpha}$ for i odd, which for each i have the same (anti-)commutators as those for i even. Under the transformations generated by these operators, $b^{\dagger}_{ia}$, $f^{\dagger}_{i\alpha}$ (i even) transform as the fundamental (defining) representation V of gl$(n+m|n)$, $\overline{b}^{a\dagger}_{i}$, $\overline{f}^{\alpha\dagger}_{i}$ (i odd) as the dual fundamental V^* (which differs from the conjugate of the fundamental, due to the negative norms). We always work in the subspace of states that obey the constraints

$$b^{\dagger}_{ia}b^{a}_{i} + f^{\dagger}_{i\alpha}f^{\alpha}_{i} = 1 \qquad (i \text{ even}) \tag{8}$$

$$\overline{b}^{a\dagger}_{i}\overline{b}_{ia} - \overline{f}^{\alpha\dagger}_{i}\overline{f}_{i\alpha} = 1 \qquad (i \text{ odd}) \tag{9}$$

(we use the summation convention for repeated indices of types a or α). These specify that there is just one 'particle' (not a particle in the sense used elsewhere in this paper), either a boson or a fermion, at each site, and so we have the tensor product of alternating irreducible representations V, V^* as desired. In the spaces V^* on the odd sites, the odd states (those with fermion number $-\overline{f}^{\alpha\dagger}_{i}\overline{f}_{i\alpha}$ equal to one) have negative norm.

For any two sites i (even), j (odd), the combinations

$$b^{a}_{i}\overline{b}_{ja} + f^{\alpha}_{i}\overline{f}_{j\alpha} \qquad \overline{b}^{a\dagger}_{j}b^{\dagger}_{ia} + \overline{f}^{\alpha\dagger}_{j}f^{\dagger}_{i\alpha} \tag{10}$$

are invariant under gl$(n+m|n)$, thanks to our use of the dual V^* of V. The TL generators acting on sites $i, i+1$ with i even are then defined as

$$e_i = \left(\overline{b}^{a\dagger}_{i+1}b^{\dagger}_{ia} + \overline{f}^{\alpha\dagger}_{i+1}f^{\dagger}_{i\alpha}\right)\left(b^{a}_{i}\overline{b}_{i+1,a} + f^{\alpha}_{i}\overline{f}_{i+1,\alpha}\right) \tag{11}$$

while for i odd

$$e_i = \left(\overline{b}^{a\dagger}_{i}b^{\dagger}_{i+1,a} + \overline{f}^{\alpha\dagger}_{i}f^{\dagger}_{i+1,\alpha}\right)\left(b^{a}_{i+1}\overline{b}_{ia} + f^{\alpha}_{i+1}\overline{f}_{i\alpha}\right). \tag{12}$$

In this case the model makes sense, and the e_i's obey the TL relations, for all integer m, though the e_i's are m times the projector onto the singlet in $V \otimes V^*$ only for $m \neq 0$ (the situation of interest in this paper). The TL algebra relations depend only on m, not n, so we have infinitely many distinct representations for each m. These representations are naturally connected with the graphical representation of the TL algebra shown in figures 1 and 2, if we view states in the representation V as flowing along the lines and remaining unchanged as they do so. Because of the sl$(m+n|n)$-invariant couplings, the lines can be consistently oriented so that, say, V flows in the direction along the arrow, or V^* in the reverse direction, as illustrated in figure 3.

Using any of these representations of the TL algebra, we may again consider the Hamiltonian (6), or alternatively the transfer matrix (3). The partition function is then that of the Potts model with $Q = m^2$. There is a transition at the self-dual parameter values, which is first order for $m > 2$, and second order for $m \leqslant 2$. The complete exact spectra of the critical theories for $-2 \leqslant m \leqslant 2$ have been determined [29]. In particular, the $m = 1$ case gives a sequence of models that represent percolation, without an ill-defined limit $Q \to 1$ being required.

There are nonlinear sigma models that are closely related to the quantum spin chains/vertex models just discussed. These have target manifolds $\mathbf{CP}^{m-1} \cong \mathrm{U}(m)/\mathrm{U}(1)\times \mathrm{U}(m-1)$, or super-manifold $\mathbf{CP}^{m+n-1|n} \cong \mathrm{U}(m+n|n)/\mathrm{U}(1)\times \mathrm{U}(m+n-1|n)$. The actions of these models contain a coupling constant g_σ and θ, the coefficient of the topological (instanton) term. For $m > 0$, the renormalization group (RG) flow of g_σ at weak coupling is towards strong coupling. A phase transition occurs at $\theta = \pi$ (mod 2π). At $\theta = \pi$ the RG flow is towards a nontrivial critical theory (conformal field theory) if $0 < m \leqslant 2$. A different behaviour occurs for m in the range $-2 < m \leqslant 0$. In both the spin chains/vertex models and sigma models, the exponents in the critical theories are independent of n for each m. For $m > 2$ there are arguments that the transition is first order, as originally found using the $1/m$ expansion [30]. We refer to [29] and references therein for the detailed arguments on all these theories and their transitions.

The quantum spin chains discussed above can be generalized to larger representations, labelled by their 'spin' $\mathcal{S}$, with $\mathcal{S} = 1/2$ in the original model. In the construction above, this can be done by replacing 1 on the right-hand side of equations (8) and (9) by the integer $2\mathcal{S}$, and using the same Hamiltonian (6) in terms of the e_i's, which are given by the same expressions, but no longer obey the TL algebra. This Hamiltonian is again essentially the Heisenberg nearest-neighbour coupling, now for all $\mathcal{S}$. As $\mathcal{S} \to \infty$ a semiclassical mapping can be made onto the corresponding nonlinear sigma models as just described [31]. In this mapping, staggering the nearest-neighbour coupling, as for $\epsilon \neq 1$, changes the value of θ in the sigma model [32]. One then argues that the spin chains have transitions, when, e.g. $2\mathcal{S}$ is odd and the nearest-neighbour couplings are unstaggered, and for $m \leqslant 2$ these are second order and all in the same universality class for each m, n. Hence the results at $\mathcal{S} = 1/2$ (the V, V^* models described above) apply to the critical points in all the spin chain models, and also in the corresponding nonlinear sigma models as θ passes though π [29, 32]. Thus the critical theories are closely related to those in the Potts model at $Q = m^2$, m integer. For the case $m = 2, n = 0$, all this reduces to the well-known results that the O(3) sigma model at $\theta = \pi$ is critical, the conformal theory is the $SU(2)_1$ WZW theory as in the spin-1/2 chain, and the S-matrices for the integrable perturbation by the $p \to \infty$ limit of the Φ_{21} operator are that of the sine-Gordon model at the coupling $\beta^2 = 2\pi$ [1] (which is equivalent to the $SU(2)_1$ WZW model perturbed by the operator tr g).

3. The Potts *S*-matrix, algebraically

The minimal models perturbed by the ϕ_{21} operator were argued to be integrable in [24]. These field theories are thermodynamically equivalent to the field theory describing the Potts models near (but off) its critical point at the special values of Q in (7). The integrability was argued to persist for all values of Q in [14].

Once it is known a model is integrable, a variety of techniques can be used to compute physical quantities. To do many such computations in an integrable field theory, it is necessary to know the S-matrix. In the S-matrix formulation, instead of dealing with a two-dimensional classical finite-temperature model, or Euclidean quantum field theory, we instead deal with a (1+1)-dimensional quantum field theory obtained by continuing one of the coordinates of two-dimensional spacetime to imaginary values. It is almost always simple to continue any physical quantity computed back and forth. The S-matrix describes the scattering of massive particles in this Lorentz-invariant quantum field theory. The theory is Lorentz-invariant because the original lattice model is rotationally-invariant in the continuum limit.

At integer values of Q, the Potts S-matrix is well understood. The S-matrix of the Ising model ($Q = 2$) for $\beta = \beta_c$ is trivial: the only particle in the spectrum has $S = -1$. The particles for the three-state model form a doublet under the S_3 symmetry; their S-matrix is diagonal and was found in [33]. The S-matrix of the four-state model that of sine-Gordon model at $\beta^2 = 2\pi$, whose S-matrix is given in [1]. In this section we discuss the Potts S-matrix at arbitrary values of Q.

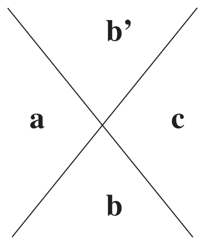

Figure 4. Representing kink scattering by four vacua.

3.1. Chim–Zamolodchikov form

A unified and physically appealing way of understanding the Potts S-matrices with $Q = 2, 3, 4$ is given in [15]. In the low-temperature phase ($\beta > \beta_c$), the Potts model has Q degenerate vacua, and the S_Q symmetry is spontaneously broken. (We consider the dual high-temperature phase at the end of this subsection.) One should be aware that at small non-zero temperature, the thermodynamic pure phase consists mainly of Potts spins in one of the Q states, with typically small domains within which one of the other $Q - 1$ spin states occurs. Each of these Q symmetry breaking ensembles of spin configurations is referred to as a vacuum in the quantum field theory point of view. The elementary *excitations* of this quantum field theory consist of kinks [15], at which location in space at a fixed time the vacuum changes from one to another of the set of Q. The worldlines of the kinks are thus domain walls in spacetime, but note that in this sense, these domain walls are *not* present in the vacua, if we assume a large system.

A typical excited state of the quantum field theory consists of regions of the various vacua, separated by domain walls. The state at a given time consists of a series of vacua $abcd\ldots$, where $a, b, c, d = 1, \ldots, Q$ with the requirement that $a \neq b, b \neq c, c \neq d, \ldots$. This configuration can then be described as a set of kinks

$$|K_{ab}(u_1)K_{bc}(u_2)K_{cd}(u_3)\ldots\rangle.$$

The kinks are assumed to be in sequence $1, 2, \ldots,$ from left to right in position space. The variables u_i are the rapidities of the various kinks, defined in terms of the energy and momentum of the kink as $E = M\cosh u$ and $P = M\sinh u$, where M is the mass of the kink. One can say that there are then $Q(Q - 1)$ different kinds of kinks, K_{ab}, with $a \neq b$ and a the vacuum to the left of the kink, b the vacuum to the right. However, it is more useful to say that there are just $Q - 1$ different kinds of particles, labelled by the possible differences $a - b (\mathrm{mod}\ Q) = 1, \ldots, Q - 1$. For a given vacuum a, one has $Q - 1$ choices of what b is. Hence the number of N-kink configurations for a at the left fixed is $(Q - 1)^N$. The $K_{ab}(u)$ can be thought of as (non-local) creation operators applied to the vacuum, say a, the vacuum at infinity at the left. We emphasize that the labelling of the kinks remains sequential in scattering events, though the indices can change, and the rapidities of two colliding particles are interchanged (as usual).

Lorentz invariance requires that any two-particle S-matrix elements be a function not of u_1 and u_2 separately, but only of the combination $u \equiv u_1 - u_2$. As a consequence of the S_Q symmetry, the S-matrix can be written in terms of four matrices A, B, C and D [15]

$$S_{\mathrm{CZ}}(u) = f^{(3)}(u)A + f^{(2)}(u)B + f^{(1)}(u)C + f^{(0)}(u)D. \tag{13}$$

The matrix elements are labelled by four indices running from 1 to Q, the four vacua involved in the scattering process, as in figure 4. The in state is $|abc\rangle$ and the out state is $|a'b'c'\rangle$, but the matrix elements always vanish unless $a = a'$, $c = c'$. The matrices A and C are diagonal, namely

$$A_{a'b'c',abc} = \delta_{ac}\delta_{bb'}\delta_{aa'}\delta_{cc'} \qquad C_{a'b'c',abc} = (1-\delta_{ac})\delta_{bb'}\delta_{aa'}\delta_{cc'}. \tag{14}$$

The non-diagonal ones are

$$B_{a'b'c',abc} = \delta_{ac}(1-\delta_{bb'})\delta_{aa'}\delta_{cc'} \qquad D_{a'b'c',abc} = (1-\delta_{ac})(1-\delta_{bb'})\delta_{aa'}\delta_{cc'}. \tag{15}$$

It is understood that all neighbouring vacua must be different (otherwise the particles would not be kinks). Thus any matrix element with $a = b$, $b = c$, $a = b'$, or $c = b'$ vanishes. The identity matrix I is $A + C$.

CZ find the functions $f^{(i)}(u)$ by demanding that the S-matrix satisfy the YB equation, and the constraints of crossing and unitarity. Here we rewrite this result in a different form. We define the matrices E and X by

$$E = A + B \qquad X = A + B + C + D \tag{16}$$

and write

$$S_{\mathrm{CZ}}(u) \propto g(u)I + h(u)E + X. \tag{17}$$

The proportionality $\propto$ here denotes equality up to a numerical function of u; this factor will not be of interest in this paper. Note that after allowing for this factor, the new parameterization still contains one less independent function than that in equation (13). The solution found by CZ does have this form, and we hope to illuminate the reason for this in our analysis. The functions $g(u)$ and $h(u)$ are found by requiring that this S-matrix satisfy the YB equation. The YB equation applies to three-particle states, labelled by $|abcd\rangle$. We denote the two-particle S-matrix acting on the two-particle state $|abc\rangle$ as S_1, and that acting on two-particle state $|bcd\rangle$ as S_2. The YB equation is then the matrix equation

$$S_i(u_1-u_2)S_{i+1}(u_1-u_3)S_i(u_2-u_3) = S_{i+1}(u_2-u_3)S_i(u_1-u_3)S_{i+1}(u_1-u_2).$$

To solve the YB equation, we need to use various algebraic relations satisfied by E and X. Using their definitions, it is simple to show that

$$\begin{aligned}(E_i)^2 &= (Q-1)E_i\\ (X_i)^2 &= (Q-2)X_i + E_i\\ X_iE_i &= (Q-1)E_i\end{aligned} \tag{18}$$

After a little work, one also finds that

$$\begin{aligned}E_iE_{i+1}E_i &= E_i\\ X_iE_{i+1}X_i &= X_{i+1}E_iX_{i+1}\\ E_iX_{i+1}E_i &= (Q-1)E_i\\ X_iE_{i+1}E_i &= X_{i+1}E_i\\ X_iX_{i+1}E_i &= (Q-2)X_{i+1}E_i + E_i\\ X_iX_{i+1}X_i - X_{i+1}X_iX_{i+1} &= X_{i+1}E_i + E_iX_{i+1} + X_i - E_i - X_iE_{i+1} - E_{i+1}X_i - E_{i+1} + X_{i+1}.\end{aligned} \tag{19}$$

All relations also hold with i and $i+1$ interchanged, and with the order of products in each term reversed. All generators here and in the rest of the paper obey $A_i B_j = B_j A_i$ when $|i-j| > 1$. The relations involving only E_i and E_{i+1} are those of the TL algebra, see equation (1), but with m replaced here by $Q-1$. Below we will see how the above algebra is related to the SO(3) BMW algebra.

By using the algebraic relations (18) and (19), the YB equation reduces to a set of functional equations for g and h. One important thing to note is that to derive the functional equations, one needs only to use the algebraic relations, and not the explicit representations of X and E. Since Q appears only as a parameter in the algebra, it appears only as a parameter in these functional equations, which can then be solved for any Q. These functional equations have been solved not only in [15], but as we will explain below, in many other contexts as well. The solution is

$$g(u) = \frac{\sinh(\lambda u - 2\mathrm{i}\gamma)}{\sinh(\lambda u)} \qquad h(u) = \frac{\sinh(\lambda u - \mathrm{i}\gamma)}{\sinh(\lambda u - 3\mathrm{i}\gamma)} \tag{20}$$

where γ is defined by

$$Q = 4\sin^2\gamma \tag{21}$$

while λ is unconstrained by the YB equation. The functions CZ use are then easily found by rewriting X and E in terms of A, B, C and D; this yields

$$g+1 = \frac{f^{(1)}}{f^{(0)}} \qquad h+1 = \frac{f^{(2)}}{f^{(0)}} \qquad g+h+1 = \frac{f^{(3)}}{f^{(0)}}.$$

The overall prefactor is found by requiring that the S-matrix be unitary and crossing-symmetric, as well as demanding consistency under the bootstrap. These conditions are discussed in detail in [15, 34]. For Potts models, one finds for example [15]

$$\lambda = \frac{3}{\pi}\gamma.$$

Another convenient way of writing this S-matrix is in terms of 'projectors' $\mathcal{P}^{(t)}$, where $t = 0, 1, 2$. We recall that an idempotent in an algebra is an element, say p, that obeys $p^2 = p$, and we say two idempotents p_1, p_2 are transversal if $p_1 p_2 = p_2 p_1 = 0$. A set of transversal idempotents that sum to the identity are projectors. Note that these definitions make no use of any inner product on the vector space, and are entirely algebraic. Thus we have $\mathcal{P}^{(t)}\mathcal{P}^{(t')} = \delta_{t,t'}\mathcal{P}^{(t)}$. Here 0, 1, and 2 stand for the spin-0, -1, and -2 representations of $U_q(\mathrm{sl}_2)$, the connection with which will be explained below. We define

$$\begin{aligned}
\mathcal{P}^{(0)} &= \frac{1}{Q-1}(A+B) = \frac{1}{Q-1}E \\
\mathcal{P}^{(1)} &= \frac{1}{Q-2}(C+D) = \frac{1}{Q-2}(X-E) \\
\mathcal{P}^{(2)} &= I - \mathcal{P}^{(0)} - \mathcal{P}^{(1)}.
\end{aligned}$$

Substituting these operators into the CZ S-matrix, one finds after some algebra that

$$S_{\mathrm{CZ}}(u) \propto \mathcal{P}^{(2)} - \frac{\sinh(\lambda u + 2\mathrm{i}\gamma)}{\sinh(\lambda u - 2\mathrm{i}\gamma)}\mathcal{P}^{(1)} - \frac{\sinh(\lambda u + 3\mathrm{i}\gamma)}{\sinh(\lambda u - 3\mathrm{i}\gamma)}\mathcal{P}^{(0)}.$$

The unitarity of the S-matrices we discuss in this paper is equivalent to $S(u)S(-u) = I$. The preceding relation makes this matrix equation reduce to a functional relation for the prefactor.

Finally, we should mention the dual description. The above kink description is very natural in the low-temperature phase. In the high-temperature phase, the vacuum is S_Q

invariant. Because of the symmetry, the fields or particles of the model should at least include a set transforming in the standard $Q-1$-dimensional representation of S_Q (such a field appears in the Landau–Ginzburg theory of the Potts model). This dimension is the same as that for a kink in the low-temperature phase. This observation provides the starting point for writing down the S-matrix for these particles in this massive phase. In this case, the matrices that take the place of E, X for a pair of particles must be S_Q invariant. Accordingly, they should be constructible within the TL algebra of such matrices in the original Potts representation, acting on the Q states for a Potts spin. The exact formulae, which involve a projection that cuts the space for these particles down to $Q-1$ dimensions, will be given in section 3 below. An alternative point of view is that we can describe the same low-temperature phase, but in terms of the dual Potts spins, by interchanging the forms of the Potts e_i's for i even and odd in equation (5).

3.2. Smirnov form

With all the above results on the CZ S-matrix, it is now straightforward to prove that Smirnov's S-matrix [14] can be written in terms of the same algebra. Precisely, we will show that Smirnov's S-matrix also can be written in the form (17), where X and E obey relations (18) and (19). As discussed in the introduction, for the S-matrices as for the transfer matrix of a lattice model, many physical quantities are the same when the same algebra appears, albeit in different representations. This idea was described and demonstrated for the S-matrices of the Φ_{13} perturbations of the minimal models in [7]. There it was shown how the S-matrix for the $O(N)$ lattice model given in [8] was in this sense equivalent to the traditional RSOS S-matrix of [10]. Thus the content of this subsection essentially consists of using the method of [7] on the Potts/Φ_{21} S-matrices. This method easily extends to the Φ_{12} S-matrices as well.

Smirnov finds a representation of the S-matrix for all Q, and also shows that when (see equation (21))

$$\gamma = \pi \frac{p-1}{2(p+1)}$$

there is an RSOS kink representation of E and X which results in a unitary S-matrix. At these values of Q (given by (7)) for p an integer, the Potts model is thermodynamically equivalent to a minimal model perturbed by the Φ_{21} operator. In [14], it is argued that these theories can be viewed as 'restrictions' of a field theory with symmetry under the quantum affine algebra $U_q\big(A_2^{(2)}\big)$. In our conventions, the parameter q of [14]) is

$$q = -\mathrm{e}^{2\mathrm{i}\gamma}$$

so that

$$Q = (q^{1/2} + q^{-1/2})^2.$$

The starting field theory is non-unitary, but after 'restriction' or 'truncation', which amounts to taking the quotient by null vectors in the non-unitary representations, the quotient theory is argued to be unitary. It still possesses the $U_q\big(A_2^{(2)}\big)$ symmetry.

The $U_q\big(A_2^{(2)}\big)$ symmetry algebra contains a $U_q(A_1)$ (i.e., $U_q(\mathrm{sl}_2)$) subalgebra. In the unrestricted case, the space of internal states for one particle is three-dimensional, and corresponds to the fundamental representation of A_2, or the spin-1 representation of $U_q(\mathrm{sl}_2)$. The particles in the restricted theory are known as RSOS kinks. An RSOS kink transforms as (a quotient of) the spin-1 representation of $U_q(\mathrm{sl}_2)$, or the corresponding $U_q\big(A_2^{(2)}\big)$ representation. The spin-s representations of $U_q(\mathrm{sl}_2)$ are similar to those of ordinary sl_2, except that when q is a root of unity, representations with $2s \geqslant p+1$ are reducible. It is

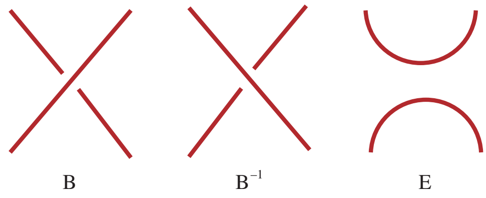

Figure 5. The generators of the SO(3) BMW algebra.

simplest to first discuss RSOS kinks in the spin-1/2 representation, which appear for example in the Φ_{13} perturbations of the minimal models [10]. Spin-1/2 kinks interpolate between *adjacent* wells of a potential with p minima when $q^{p+1} = 1$. In other words, numbering the vacua $a = 1, 2, \ldots, p$, the kinks are of the form $|ab\rangle$, where a, b are in $1, \ldots, p$, and $b = a \pm 1$. (This is as distinguished from the Potts kinks $|ab\rangle$ with $a \neq b$.) Spin-1 kinks have the same allowed vacua, but with a different requirement. They are of the form $|ab\rangle$ where a, b are in $1, \ldots, p$, and $b = a \pm 2$, or $|aa\rangle$ for any $a = 2, \ldots, p-1$. One can count the 'number' K of distinct particle states: while the number of N-kink states is an integer for all N, as $N \to \infty$, it behaves as K^N. Here it is straightforward to show that (following, e.g. [35])

$$K = 4\cos^2\left(\frac{\pi}{p+1}\right) - 1 \leqslant 3.$$

Note that $K = Q - 1$, just like the Potts kinks described above, however, here we have a theory that makes sense for any Q such that p is an integer, not just $Q = 1, 2, 3, 4$. For $Q = 2$, 3, 4, this construction reproduces the Potts model construction of CZ.

The S-matrix (for both restricted and unrestricted theories) is of the form

$$S_{\mathrm{Sm}}(u) \propto (\mathrm{e}^{-2\lambda u} - 1)\,\mathrm{e}^{3\mathrm{i}\gamma} B + (\mathrm{e}^{2\lambda u} - 1)\,\mathrm{e}^{-3\mathrm{i}\gamma} B^{-1} + 4\sin(3\gamma)\sin(2\gamma) \qquad (22)$$

where B and B^{-1} are u-independent matrices, and $BB^{-1} = B^{-1}B = I$. In the unrestricted case, B and B^{-1} are nine-dimensional. For both the unrestricted and restricted (p integer) cases, the explicit forms can be found in [14]. In both cases, B and B^{-1} are associated with the constant (u-independent) solution of the YB equation for particles in the spin-1 representation of $U_q(\mathrm{sl}_2)$. This is why the projectors discussed above are those of the spin-0, 1 and 2 representations, the representations which appear in the tensor product of two spin-1 representations of $U_q(\mathrm{sl}_2)$. However, S_{Sm} is not the standard solution of the (rapidity-dependent) YB equation with spin-1 particles and $U_q(\mathrm{sl}_2)$ symmetry; it is instead associated with the fundamental representation of $U_q\big(A_2^{(2)}\big)$ [36, 37]. (The standard $U_q(\mathrm{sl}_2)$-invariant S-matrix with spin-1 particles will be considered in section 6.)

Now our task is to show that the S-matrix given by (22) is equivalent to the S-matrix (17), with the functions g and h given by (20). We do this by relating B and B^{-1} to X and E. It was noted in [38] that B and B^{-1} satisfy the SO(3) BMW algebra [3]. This algebra is usually written in terms of B and a TL generator E, where B^{-1} is related to E via

$$B^{-1} = B + (q - q^{-1})(E - I).$$

Pictorially, one can represent B, B^{-1} and E in figure 5; B is a braiding operation, while E is proportional to an idempotent (for $Q \neq 0$). The generators E_i then obey the TL algebra (the first relation in each of equations (18) and (19)), where indeed $Q = (q^{1/2} + q^{-1/2})^2$.

The remaining relations of the SO(3) BMW algebra are

$$\begin{aligned}
B_i E_i &= q^{-2} E_i \\
B_i B_{i+1} B_i &= B_{i+1} B_i B_{i+1} \\
B_i E_{i+1} B_i &= B_{i+1}^{-1} E_i B_{i+1}^{-1} \\
B_i B_{i+1} E_i &= E_{i+1} B_i B_{i+1} = E_{i+1} E_i \\
B_i E_{i+1} E_i &= B_{i+1}^{-1} E_i \\
E_i B_{i+1} E_i &= q^2 E_i
\end{aligned} \tag{23}$$

and those relations given by interchanging i and $i+1$, as well as reversing the order of both sides. It is now simple to show that these relations are identical to (18) and (19), if

$$\begin{aligned}
B &= qI - X + q^{-1} E \\
B^{-1} &= q^{-1} I - X + qE.
\end{aligned} \tag{24}$$

Plugging these into (22), one indeed recovers (17) and (20).

We now have a unified form of the S-matrix for the scaling limit of the Q-state Potts model in the region $Q \leqslant 4$. It is given by equation (17) or (22), in terms of generators of the SO(3) BMW algebra and thus, as an element in the algebra, independent of any representation. The YB equation has been verified, and the functions in equation (17) or (22) found, using only the relations in the algebra. Consequently, by using a representation of the BMW algebra, in which the generators become matrices, we obtain the S-matrix appropriate for that representation (up to overall functions independent of the representation). There are four separate representations for this S-matrix, with two more to be described in the next section. The first two we discussed are the low-temperature Potts representation [15], and its dual, the high-temperature Potts representation; both are valid for $Q = 2, 3$ and 4. The third is the unrestricted $U_q\big(A_2^{(2)}\big)$-invariant S-matrix, valid for all $Q \leqslant 4$. Finally, the fourth is the restricted $U_q\big(A_2^{(2)}\big)$ or RSOS kink representation, valid for p integer $\geqslant 3$, where $Q = 4\sin^2\gamma$ and $\gamma = \pi(p-1)/[2(p+1)]$ [14]. The last two versions are self-dual, in the sense that the matrices for e_i are the same for i even and odd.

We have again omitted the overall function multiplying the S-matrix; this is determined by crossing, unitarity and the bootstrap; it can be found in either [14] or [15] (some subtleties regarding the bound states are discussed in [34]). Using the fact that E and X are Hermitian (with respect to the appropriate inner product) in all the representations we discuss, all these conditions can be reduced to functional equations by using only the algebraic relations. For $Q \leqslant 3$, the particles discussed so far (corresponding to spin-1 kinks in $U_q(\mathrm{sl}_2)$) are the only ones, while for $3 < Q \leqslant 4$, the spectrum also contains a stable singlet particle (i.e. corresponding to spin 0 in $U_q(\mathrm{sl}_2)$). There should be no difficulty in writing down the spin 0–0, 0–1, and 1–0 S-matrices, and verifying the YB equation, in terms of the BMW algebra also. By redefining the parameters q and γ as described in [14], all of the above considerations also apply to the Φ_{12} perturbations of the minimal models of conformal field theory. In the language of lattice statistical mechanics, these correspond to the scaling limit of the tricritical Q-state Potts models.

4. The Potts *S*-matrix in TL language and the sigma models

In this section we provide yet another way of representing the Potts S-matrix. This representation is entirely in terms of TL generators. There are several reasons why this is useful. It is somewhat simpler to deal with the TL algebra only, instead of the much more

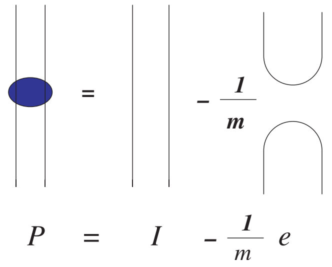

Figure 6. The projection onto the adjoint.

complicated BMW algebra. This provides a nice intuitive connection with the Boltzmann weights of the lattice model, which also can be expressed in terms of generators of the TL algebra. This representation also illuminates a connection to the SU(m)-invariant representation with $Q = m^2$, described in the introduction; in fact, we must generalize to sl($m + n|n$) supersymmetry in order that this be meaningful for $m = 1$. According to the arguments of [18, 29], this describes the supersymmetric sigma models on $\mathbf{CP}^{m+n-1|n}$, $m = 1$, 2, with the critical theory at $\theta = \pi$ perturbed by changing θ. Finally, it will also allow us to provide in the next section a nice interpretation of the boundary S-matrices of [23], with applications to all the above representations.

4.1. The Potts S-matrix in terms of TL generators

Rather than immediately giving the formulae for the SO(3) BMW generators in terms of the TL generators, we will begin by briefly indicating the motivation that led to them. First, we note that in the Smirnov form of the S-matrix, the possible states of a single particle (kink) were related to the spin-1 representation of $U_q(\mathrm{sl}_2)$. Spin-1 can be obtained within the tensor product of two spin-$1/2 U_q(\mathrm{sl}_2)$ representations. In iterated tensor products of spin-1/2, the TL algebra is the commutant of $U_q(\mathrm{sl}_2)$, the algebra of invariant operators in the representation, at least for generic q (see [39] for a review). Thus the projectors of two-particle states i, $i + 1$ onto spin-0 and spin-1 are given by e_i/m, $I - e_i/m$, respectively. This is displayed pictorially in figure 6. We then expect that the $U_q(\mathrm{sl}_2)$-invariant S-matrix of spin-1 particles can also be written in terms of TL generators.

To find the explicit form, it is useful to consider the graphical representation of the TL algebra, already illustrated in figures 1 and 2. Essentially, the TL algebra is the algebra of diagrams consisting of (topological equivalence classes of) non-crossing lines that join $4N$ points on a circle in pairs through the interior, the product being joining of the $2N$ points on a semicircle at the bottom of one diagram to the $2N$ on a semicircle at the top of another. Each point represents a spin-1/2 of $U_q(\mathrm{sl}_2)$. The generators e_i represent a pair of 180° turns. Including the projector for each pair of spin-1/2's to spin-1, then the scattering of kinks represented by a pair of lines should be described by the possible ways of either continuing the lines through the scattering, or joining them in 180° degree turns, but never allowing the lines to cross or join their partner. There are exactly three linearly independent ways to do this, just as there are three terms in the decomposition of the tensor product of spin-1 with itself. We arrive at the representation of the three possibilities shown in figure 7. These are (up to constant factors) I (the identity), E_i, and X_i defined earlier (and this was the reason for those definitions). The projectors are linear combinations of I and e_i's, but do not introduce crossings either. This construction has been known for some time [4, 5, 40].

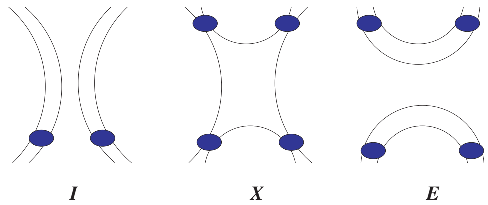

Figure 7. The generators of the BMW algebra in terms of the TL algebra.

The preceding pictures translate into the following expressions. In terms of the projectors $\mathcal{P}_i = I - e_i/m$ onto spin-1, we have

$$\begin{aligned} E_i &= \mathcal{P}_{2i+1}\mathcal{P}_{2i+3}e_{2i+2}e_{2i+1}e_{2i+3}e_{2i+2}\mathcal{P}_{2i+1}\mathcal{P}_{2i+3}, \\ X_i &= m\mathcal{P}_{2i+1}\mathcal{P}_{2i+3}e_{2i+2}\mathcal{P}_{2i+1}\mathcal{P}_{2i+3}. \end{aligned} \tag{25}$$

Using only the TL algebra relations (1), one can show (after some effort) that the generators E_i and X_i defined by (25) satisfy relations (18) and (19) [40]. In this proof, a very useful set of identities is

$$E_i = \mathcal{P}_{2i+2\pm1}e_{2i+2}e_{2i+1}e_{2i+3}e_{2i+2}\mathcal{P}_{2i+2\pm1} = \mathcal{P}_{2i+2\pm1}e_{2i+2}e_{2i+1}e_{2i+3}e_{2i+2}\mathcal{P}_{2i+2\mp1}.$$

This identity is obvious in the graphical language, but takes some work to prove using the TL algebra alone.

Thus from any representation of the TL algebra we can find using equation (25) a representation of the SO(3) BMW algebra, and hence of the Potts S-matrix. The original lattice Potts model representation (5) of the TL algebra yields the low-temperature Potts kink S-matrix of CZ, while the same Potts representation but with -1 added to the indices on the right-hand side in equation (25) reproduces the dual high-temperature Potts S-matrix mentioned at the end of subsection 2.1. In these cases, the projectors $\mathcal{P}_{2i}$ or $\mathcal{P}_{2i+1}$ are onto a space of $Q-1$ states for each particle. The six-vertex/XXZ representation of the TL algebra should result in the unrestricted $U_q\big(A_2^{(2)}\big)$-invariant S-matrix, while the RSOS spin-1/2 kink representation of the TL algebra should result in the RSOS spin-1 kink S-matrix given in [14], as discussed in subsection 2.2.

4.2. The S-matrices for the supersymmetric sigma models

The final examples we want to discuss in a little more detail are the supersymmetric sigma models with $\mathrm{sl}(m+n|n)$ supersymmetry. We find here the S-matrix describing the massive field theory given by perturbing the topological angle θ away from π. All values of θ will be mod 2π here; later we discuss the more general situation. This S-matrix is valid for $m = 1, 2, n > 0$ (the cases $n = 0$ are trivial and known, respectively), and the related lattice models. In the appendix we will discuss the situation with $m > 2$.

A useful starting point is the result in the $1/m$ expansion of the $\mathbf{CP}^{m-1}$ sigma models [30] (also valid for the $\mathrm{sl}(m+n|n)$-invariant extensions) that the transition for bringing θ through π is first order. The first-order transition point is believed to persist for all $m > 2$. At the transition point, there are two competing phases (vacua) of the same energy density (energy per unit length), that correspond to, say $\theta = \pi - 0^+$ and $\pi + 0^+$, respectively, where 0^+ represents a positive infinitesimal. Elementary excitations are domain walls (kinks) separating these phases, which are simply the boson excitations or 'spinons' of the theory, that transform in the fundamental V of SU(m), and the dual V^* for the antiparticles. For the kinks in the fundamental, we have the $\theta = \pi - 0^+$ vacuum to the left of it, and the $\theta = \pi + 0^+$ vacuum to the right; the reverse applies to the dual fundamental. Thus if kinks are present, they must alternate in type along the system. Otherwise there will be a region of yet another phase in between, with a larger energy density. Off the transition point $\theta = \pi$, the energy density for the two phases is different. Then the kinks are confined in pairs by a linear potential, and transform in the representation $V \otimes V^*$ of SU(m) or sl($m+n|n$). In the phase θ slightly $<\pi$, the pair has V to the left of V^*, and the reverse in the phase θ slightly $>\pi$. For $0 < m \leqslant 2$, the transition is second order, but the picture of the phases off the transition remains valid. The field theories we construct via their S-matrices describe either of these two phases, in the scaling limit near $\theta = \pi$. Thus at short distances, the running value of θ approaches π, and the theory approaches the critical (conformal) field theory or RG fixed-point theory of the transition itself.

The argument just given for the form of the confined pairs of kinks off the transition is not rigorous as it stands, because it is not clear that small regions of vacua other than the two on either side of the transition (formed by allowing the kinks to cross, or by creating a pair of kinks in the 'wrong' ordering) can really be neglected. It appears strongest in the second-order case, where we may focus on low energies, and the spinons are massless (gapless) at the critical point, but could fail completely in the first-order case. To reinforce the argument for the second-order case, we appeal to the lattice models, the six- and supersymmetric- vertex models, or quantum spin chains. As mentioned in subsection 2.3, these are argued to be in the same universality classes as the sigma models. In the spin chain models (see equation (6)), the two phases are at $\epsilon = 1 + 0^+$ and $1 - 0^+$ (our conventions imply that $\theta = \pi - O(\ln \epsilon)$). In the ground state for $\epsilon > 1$, the expectation $\langle e_i \rangle$ for i even is larger than for i odd, and *vice versa* for $\epsilon < 1$. This phenomenon is known as dimerization or spin-Peierls ordering, especially when it occurs spontaneously, breaking the reflection symmetry of the chain about a site present when $\epsilon = 1$. Such spontaneous symmetry breaking occurs in these models only for $m > 2$, implying that the transition as ϵ passes through 1 is first order. The phase when $\epsilon > 1$ can be pictured as represented by the state in which pairs of sites $i, i + 1$ with i even form singlets. This is an eigenstate of H only when ϵ goes to ∞, but is a useful picture generally. The other phase, the ground state for $\epsilon < 1$, is represented by dimerizing the other way, as singlet pairs $i, i + 1, i$ odd. Then a kink between these two phases must consist of a site that is not forming a singlet pair with either neighbour, and so carries the representation V or V^* exactly as for the sigma model described above. This conclusion is exact, and not just an artefact of the simple picture of the two phases. In the time evolution governed by H, the action of the TL generators e_i ensures that the unpaired sites never cross; their worldlines are simply the lines in the graphical representation of the TL algebra that we have already used. Thus the excitations in the critical theory also consist of kinks that must alternate in type along the system, and this is in fact implicit in the results in [29]. Off criticality, the kinks, which correspond to the spin-1/2 kinks of the $U_q(\mathrm{sl}_2)$ representation, are confined in pairs to form the spin-1 kinks.

These arguments mean that in the continuum (scaling) limit, the S-matrix of these spin-1 kinks in the sigma model off of $\theta = \pi$ is of the form (17). The matrices E_i, X_i are written in terms of the TL generators as in (25), and here the TL generators are in the supersymmetric representation given by (11) and (12). To repeat what was stated in subsection 2.3 [18, 29]: in terms of the alternating spaces V, V^* for the kinks (with V for i even, V^* for i odd), e_i/m is the projector onto the singlet in $V \otimes V^*$ (or $V^* \otimes V$) for neighbours $i, i + 1$. The expressions in equation (25) are for the phase $\theta = \pi + 0^+$, or $\epsilon < 1$, while those for the other phase $\theta = \pi - 0^+$, or $\epsilon > 1$ are again obtained by adding -1 to the indices on the e_i's and $\mathcal{P}_i$'s. We note again that for $m < \sqrt{3}$ (i.e. $m = 1$!), the singlet part of a bound kink pair in $V \otimes V^*$ (or $V^* \otimes V$) does not correspond to a stable particle in the spectrum.

5. The boundary Potts S-matrix, algebraically

The scaling limit of the Potts model remains integrable in the presence of certain boundary conditions on the half-plane. In the language of Potts spins, these are free and fixed boundary conditions, and we again focus on the low-temperature phase. 'Free' means the Potts spin is unconstrained at the edge, and the model has the full S_Q symmetry (though this is broken spontaneously in the low temperature phase). 'Fixed' means that the value of the Potts spin at the edge is given, and the same all along the edge. In this case, the symmetry is broken explicitly to S_{Q-1}. The corresponding boundary S-matrices were found in [23]. A boundary S-matrix in an integrable theory must satisfy a variety of constraints, including the boundary version of the YB equation [41]. The boundary YB equation is

$$R(u_2)S_i(u_1+u_2)R(u_1)S_i(u_1-u_2) = S_i(u_1-u_2)R(u_1)S_i(u_1+u_2)R(u_2)$$

where $i = 0$ or $N-2$ refers to the bulk S-matrix for the two bulk particles closest to the boundary at the end. In the case at hand, the boundary YB equation gives the overall form of R, the boundary S-matrix. Since a solution of the boundary YB equation can be multiplied by any function, one must use the remaining conditions (the boundary analogues of the crossing, unitarity and bootstrap conditions) to fix this function. In this section we show how to formulate the two boundary S-matrices algebraically, so that the boundary S-matrix applies to all the representations of the Potts S-matrix.

The boundary S-matrix for fixed boundary conditions is proportional to the identity matrix [23], and so is trivial to describe algebraically. The reason for this is easy to see in the language of Potts spins: since the spin on the boundary remains fixed, a kink approaching the boundary must merely bounce off. Note that in this case, with the Potts spin at one end of the system fixed, there are exactly $(Q-1)^N$ states for N kinks with given rapidities. This space of states is a subspace (determined by the projectors $\mathcal{P}_{2i+1}$, $i = 0, \ldots, N-1$) of the states in which the e_i ($i = 0, \ldots, 2N$) act once more in the Potts representation (5), and the bulk S-matrices are given in terms of E_i, X_i, by equation (25), with $i = 0, \ldots, N-2$ for N kinks. We note that use of this representation of the e_i's implies that the total space contains Q^{N+1} states, of which $Q(Q-1)^N$ survive projection, and the factor of Q means that all possible boundary spin values are in fact present, though they do not mix in the scattering.

The boundary S-matrix for free boundary conditions is not diagonal. In the language of Potts kinks, a kink $|ab\rangle$ scattering off the boundary at the right can scatter to any state $|ab'\rangle$, as long as $b' \neq a$. The permutation symmetry S_Q means that the boundary S-matrix can be written as

$$R_{\text{free}}(u) \propto I + r(u)Z \tag{26}$$

where in the Potts-kink representation used in [23], each element

$$Z_{a'b',ab} = \delta_{aa'}$$

i.e. in this representation, for each $a = a'$, Z is a $(Q-1)\times(Q-1)$ matrix with every entry 1. Note that in the free case, the number of states of N kinks with given rapidities is $Q(Q-1)^N$, the same as in the fixed case, but now all are mixed by the scattering processes.

This boundary S-matrix for the free case can be described in terms of the TL generators. Precisely, if the Nth kink is the last before the boundary at the right, then we have

$$Z = m\mathcal{P}_{2N-1}e_{2N}\mathcal{P}_{2N-1}. \tag{27}$$

The $\mathcal{P}_{2N-1}$ project the incoming particle and outgoing particle onto $Q-1$ states, as required. The e_{2N} then is responsible for the boundary scattering. There is a similar expression for scattering of the first kink off the boundary at the left.

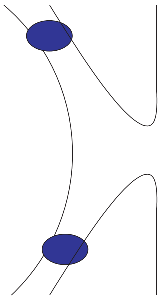

Figure 8. Boundary scattering in terms of the TL generators.

In terms of the representation of the TL algebra in terms of non-crossing lines, this can be represented as in figure 8. There is a degree of freedom on the boundary, represented by the vertical line, corresponding in the unrestricted $U_q(\mathrm{sl}_2)$ representation to the spin-1/2 representation. With one of these at each end, they contribute the factor $m^2 = Q$ to the count of the number of states. The algebraic relations of E_{N-2} and X_{N-2} with Z then follow immediately from the TL algebra (1). There are many of them, so we will not write out them all. The ones useful in solving the boundary YB equation are

$$\begin{aligned}
&Z^2 = (Q-1)Z \\
&E_{N-2}ZX_{N-2} = (Q-2)E_{N-2}Z \\
&ZE_{N-2}ZE_{N-2} = (Q-1)ZE_{N-2} \\
&ZE_{N-2}ZX_{N-2} = (Q-2)ZE_{N-2}Z + ZE_{N-2} \\
&ZX_{N-2}ZE_{N-2} = (Q^2-3Q+1)ZE_{N-2} \\
&ZX_{N-2}ZX_{N-2} = (Q-2)^2ZE_{N-2}Z + (Q-2)ZE_{N-2} + ZX_{N-2}.
\end{aligned}$$

Using these relations, it is then straightforward but tedious to plug the S-matrix and R-matrix into the boundary YB equation and find that

$$r(u) = \frac{\sinh(2\lambda u + 3\mathrm{i}\gamma)}{2\cosh(\gamma)\sinh(2\lambda u)}.$$

This is in agreement with the results of [23], where the boundary S-matrix is written in the form

$$R_{\mathrm{free}}(u) = r_1(u)I + r_2(u)(Z - I)$$

so that $r = r_2/(r_1 - r_2)$. The boundary YB equation thus determines the ratio r_1/r_2, but by using the boundary versions of crossing, unitarity and the bootstrap, the functions themselves have been determined [23].

The utility of the expression (27) is that the boundary S-matrix can now be found explicitly in any representation of interest. In addition to the Potts-kink representation of [23], there is the high-temperature version, in which -1 is added to the indices in the expressions for E_i, X_i and Z. The S-matrix for the 'fixed' boundary condition of the low-temperature case becomes that for the free boundary condition of the high-temperature case, and can be represented in a total space of only Q^N states (before application of the projectors). The S-matrix for the 'free' boundary condition of the low-temperature case becomes one with additional boundary degrees of freedom (discussed further below) in the high-temperature case, and presumably means a fixed boundary condition on the *dual* Potts variables. The $U_q(A_2^{(2)})$ representation of the boundary S-matrix was found in [42]. The RSOS-kink representation (valid for all the Φ_{21}-perturbed minimal models) and supersymmetric representation of this boundary S-matrix seem to be previously unknown. One can easily extend these considerations to the Φ_{12}-perturbed minimal models as well.

Once again, the application in the example of the $\mathbf{CP}^{m+n-1|n}$ sigma models deserves comment. In the presence of a boundary, the topological term in the sigma model is no longer an integer, and the physics is not periodic in θ (this and the effect of the θ term on the boundary conditions are discussed extensively in [43]). The region $-\pi < \theta < \pi$ behaves like the $\epsilon > 1$ region of the quantum spin chain with Hamiltonian (6). In this case the dimerization extends up to the end of the chain, and there is no boundary degree of freedom. On the other hand, in the region $\pi < \theta < 3\pi$, the spin chain dimerizes the other way, and an unpaired spinon is left at each end; the spinon (kink) in the representation V is bound by a linear potential to the left end of the system, and another in the dual V^* is bound at the right end (if the system is large). These occur to lower the bulk energy density of the vacuum to its lowest possible value [30]; the boundary 'knows' that the underlying θ is $> \pi$ and so a different vacuum occurs immediately adjacent to the edge, separated from the bulk by a kink. Consequently, with N particles (bound kink pairs) in the bulk, the number (i.e. dimension of the space) of states for a given set of rapidities is $(m+2n)^2[(m+2n)^2-1]^N$, similar to the Potts model with $Q = m^2$ [however, the superdimension of the space [29] is $m^2(m^2-1)^N$]. Similarly, if we go to the region $|\theta - 2\pi s| < \pi$, $s \geqslant 0$, then there will be s spinons bound at each end. For $s \leqslant 0$, the role of V and V^* at the boundary is reversed, as well as in the bulk.

Explicitly, for the scaling limit of the spin chain with V, V^* alternating, such that V is at the left end, and V^* is at the right, we find that for $\epsilon > 1$, corresponding to $\theta < \pi$, the particles in the bulk have V to the left of V^* (see end of section 3), and there is no degree of freedom on the boundary, so the forms for the high-temperature Potts phase with free boundary conditions apply. For the spin chain with the same boundary conditions as before but now with $\epsilon < 1$ ($\theta > \pi$), the particles in the bulk have V^* to the left of V, and there is a boundary spinon in V at the left, and V^* at the right. In this case the S-matrices are precisely as given in equations (25) and (27), the free boundary conditions for the low-temperature Potts phase.

We have found that the cases $s = \pm 1$, as well as $s = 0$, are integrable in the scaling limit near the transition, for $m = 2$. We do not know if the cases $|s| > 1$ are integrable (the spin chain models with larger $\mathcal{S}$ contain phases with $s > 1$ as well as $s = 0, 1$, or negative values if one spin is added or removed from each end of the chain).

It is amusing that in the application of the $\mathbf{CP}^{1|1}$ sigma model to the spin quantum Hall transition, the boundary degree of freedom present on one side of the transition is just the famous 'edge state' familiar from the related integer quantum Hall effect. Thus here we have, in principle, access to some exact properties of these states in the scaling regime near the transition.

6. Other S-matrices for supersymmetric sigma models

In the previous sections, we have obtained the S-matrices for a perturbation of the critical theory of the $\mathbf{CP}^{m+n-1|n}$ sigma models at $\theta = \pi$ by a change in θ, which corresponds to an RG flow from the critical theory at $\theta = \pi$ in the ultraviolet to the $\theta = 0$ massive fixed point in the infrared. A byproduct is that if we take the ultraviolet limit of our S-matrices, or equivalently let the mass scale M go to zero with the energies and momenta of the particles fixed, then we obtain a description of the critical theory at the $\theta = \pi$ fixed point. In fact, we obtain two inequivalent, but dual descriptions of this theory, since our particles either have the representation V to the left of V^*, or vice versa, depending which side of the transition we come from. These representations V, V^* are dual and inequivalent (except in the cases $n = 0$ and either $m = 1$, which is trivial, or $m = 2$, which is the usual $\mathrm{SU}(2)_1$ WZW theory), and hence so are our descriptions of the theory. Presumably though, they do lead to descriptions of the same theory in the ultraviolet.

Other S-matrices are known for certain sigma models. In particular, for the O(3) sigma model, there is an S-matrix for massive particles in the vector (spin 1) representation of SO(3) (or even for SO(m)), which represents the RG flow from the weak-coupling fixed point in the ultraviolet to the strong-coupling fixed point at $\theta = 0 \pmod{2\pi}$ in the infrared [1]. There is also another for *massless* spin-$1/2$ particles that represents the RG flow from weak coupling into the nontrivial fixed point at $\theta = \pi$ in the infrared [44]. This one is for the O(3) model only (the O(m) model for $m > 3$ has no θ term). Both of these possess generalizations in which the symmetry is deformed to $U_q(\mathrm{sl}_2)$. In the RSOS representations, these have been used to describe parafermion theories under an integrable perturbation [45].

For symmetry reasons, both of these S-matrices can be written in terms of the TL algebra. Thus following the work of section 4, we can use the supersymmetric representation of the TL algebra to find the supersymmetric S-matrices. This is simple to see for the massless spin-$1/2$ ($\theta = \pi$) case. Here the S-matrix is a linear combination of I and e, with u-dependent coefficients. There are different S-matrices for scattering left- with left-, right- with right-, or left- (right-) with right- (left-)-moving particles. We may now utilize these S-matrices in the supersymmetric representation of the TL algebra. The massless particles in the spectrum at $\theta = \pi$ are assumed to be kinks that are alternately in the V or V^* representations. Then it is immediate that the S-matrices for these massless kinks can be written in terms of the TL generators.

For the spin-1 representation of $U_q(\mathrm{sl}_2)$, the S-matrix can be written in terms of the generators I, E, X of the SO(3) BMW algebra. As in previous sections, we can then use the same forms (with the BMW generators written in terms of TL generators as before) for scattering of bound pairs of kinks, where the kinks must alternate, and so either every pair has V to the left of V^*, or every pair has the reverse order.

These constructions give S-matrices in supersymmetric representations that parallel those for the O(3) sigma model RG flows at $\theta = 0$ or π, respectively. It is natural to try to identify these S-matrices as describing the analogous flows in the $\mathbf{CP}^{m+n-1|n}$ sigma models, for $0 < m \leqslant 2$ where similar RG flows are expected. Certainly the types of particles present in the infrared limit seem plausible: kinks in V, V^* alternately in the $\theta = \pi$ case, and in the adjoint (part of $V \otimes V^*$) for $\theta = 0$. Actually, the latter is disturbing, since as we pointed out, there are two distinct forms, depending whether V is to the left or right of V^* in all the particles, and this appears to give two distinct S-matrices. The choice of one or other of these would break the reflection (parity) symmetry of the model at $\theta = 0$. It could be that there is an isomorphism that would show the equivalence of the two choices, for particular values of m, but this is not clear to us. (It was not an issue at all in the previous sections, where reflection symmetry was explicitly broken by $\theta \neq 0$ or π, nor for the $\theta = \pi$ case here with free boundaries, since $\theta = \pi$ and $-\pi$ are then distinct.) A check on the proposal is to calculate properties in the ultraviolet. From the thermodynamic Bethe ansatz, the central charge of the theory at the fixed point in the ultraviolet can be obtained. It is known that this is $c = 2(m-1)$ for $m = 1, 2$, for either case $\theta = 0$ or π (the calculations are thermodynamic and independent of the representation used, and hence independent of n). The correct value at the weak-coupling fixed points in the $\mathbf{CP}^{m+n-1|n}$ sigma models is always $c = 2(m-1)$, which is a useful check on our claims.

It is possible to construct spin-chain models, different from those in section 2.3, that are described in the continuum limit by these supersymmetric S-matrices. The spins have spin $\mathcal{S}$ under the $U_q(\mathrm{sl}_2)$ symmetry. To construct a Heisenberg-like coupling between nearest neighbours, we need to study their $U_q(\mathrm{sl}_2)$ representation properties. In general, two spins decompose as in sl_2, into total spins $0, 1, \ldots, 2\mathcal{S}$. This can always be done within the TL algebra, by projecting of a set of $2\mathcal{S}$ spin-1/2's onto their spin-$\mathcal{S}$ part. (In the $\mathcal{S} = 1$ case, one of course ends up with the BMW algebra, as has been discussed at length in this paper.) Then we may take a linear combination of these terms chosen so that the possible eigenvalues in each total spin sector are those of the invariant bilinear form of generators of $U_q(\mathrm{sl}_2)$, with one generator on each site. This is then the analogue of Heisenberg coupling for $U_q(\mathrm{sl}_2)$, and can be carried over to any representation of the TL algebra, such as the supersymmetric ones. With this we may then build a chain out of these 'spin-$\mathcal{S}$' representations in any representation of the TL algebra, by using this coupling for nearest neighbours. It consists of $2N$ medial-graph sites, with $2N$ divisible by $2\mathcal{S}$. Note that for $2\mathcal{S}$ odd, the spin-$\mathcal{S}$ representations must alternate with their duals, in order that the TL sites always alternate between V and V^* in the supersymmetric representation. For $2\mathcal{S}$ even, there are two different constructions, depending whether in each group of sites V or V^* is the first one at the left. We again consider only free boundary conditions. Note that the spin-$\mathcal{S}$ representations in the supersymmetric representation of TL are multiplets with the degeneracies found in the appendix of [29], which are larger than those of irreducible representations of $\mathrm{sl}(m+n|n)$ for $n > 0$, except for the cases $\mathcal{S} = 0, 1$, which are the singlet and adjoint of $\mathrm{sl}(m+n|n)$. Note also that the 'spin-$\mathcal{S}$' representations here, which are part of the product $V \otimes V^* \otimes V \cdots$ ($2\mathcal{S}$ times), (or $V^* \otimes V \otimes V^* \cdots$ ($2\mathcal{S}$ times), or these alternately if $2\mathcal{S}$ is odd), are different from the 'spin-$\mathcal{S}$' representations in section 2.3, which alternated between supersymmetrized products of $2\mathcal{S}$ copies of V and of the same for V^*.

Like the spin-$\mathcal{S}$ $U_q(\mathrm{sl}_2)$ Heisenberg chain they are related to, these lattice models are not integrable, except for $\mathcal{S} = 1/2$. (It is possible to fine tune the coefficients of the nearest-neighbour couplings to obtain integrable models, but these are multicritical and not of our interest here. Such models have been constructed by this same procedure in [46], though the supersymmetric representation was not considered.) But as $\mathcal{S} \to \infty$, a semiclassical approximation is valid, as for the SU(2) ($q = 1$) chains [31]. In such a limit, this chain will map onto some sort of nonlinear sigma model. For the supersymmetric representation of TL with $n > 0$, the target space of the sigma model is a noncommutative space, that so far has been defined only by the 'space of functions' on it, which by definition consists of the multiplets of total $\mathcal{S} = 0, 1, 2, \ldots,$ which occur in the decomposition of the product of two sites, for $2S$ either even or odd, as $\mathcal{S} \to \infty$. By construction, the eigenvalues of the Hamiltonians of these chains can be found for finite length and finite $\mathcal{S}$ using only the TL algebra, and so are independent of the representation used (though multiplicities could vanish in particular cases). In the continuum $\mathcal{S} \to \infty$ limit (taken in the obvious way, with the lattice constant going to zero, and the velocity of 'light', the mass scale M, and the length L of the system staying fixed) we then expect that these theories *are* integrable, and have no doubt that the S-matrices for all the RG flows considered in this paper, including this section, apply to these models. In particular, the lines in the graphical representation of TL never cross, as in the S-matrix constructions, and this is related to the symmetry properties in [47]. It is not obvious that there is a problem with parity symmetry, as there are two distinct models microscopically for $\theta = 0$ (we will use the same terminology as before, $\theta = 0$ and $\theta = \pi$ for the cases with no staggering of the couplings and $\mathcal{S} \to \infty$ through values $2\mathcal{S}$ even and odd, respectively).

A more difficult question is whether these S-matrices describe the $\mathbf{CP}^{m+n-1|n}$ sigma models in their flows along $\theta = 0$ or π, or in other words, whether the continuum limits of our spin chains are the same as these sigma models. In view of the different geometry of the target spaces in the semiclassical (weak-coupling) limit, this seems unlikely. In particular, we may compare the spectra of the continuum models in the weak-coupling/semiclassical regime in finite size, where the length acts as an infrared cutoff. For free boundary conditions, the lowest states are those where the sigma model field is constant along the system, and behaves as a free quantum-mechanical particle moving on the target space. For both the models defined here, and the $\mathbf{CP}^{m+n-1|n}$ sigma models, these lowest states can be labelled in increasing order by total $\mathcal{S} = 0, 1, 2$, etc, where $\mathcal{S} = 0, 1$ are the singlet and adjoint of $\mathrm{sl}(m+n|n)$, but for higher $\mathcal{S}$ the multiplicities in the models based on the TL algebra are those found in the appendix of [29], and are larger than those for motion on $\mathbf{CP}^{m+n-1|n}$. The latter can be found from the spin chain models in section 2.3, by decomposing two sites of spin-$\mathcal{S}$ (as defined there), and taking $\mathcal{S} \to \infty$. In fact, for given m, n, the multiplicities increase exponentially with $\mathcal{S}$ in the models here [29], but as powers of $\mathcal{S}$ for the $\mathbf{CP}^{m+n-1|n}$ sigma models, or the spin chain models of section 2.3. On the other hand, for a long system $L \gg M^{-1}$, the states at finite excitation energy are built from a finite number of excitations (massive particles) in the adjoint (for the $\theta = 0$ case) in both sigma models, and it is not obviously ruled out that the spectra coincide.

We should point out that the partition functions, with no supersymmetry-breaking source terms, and with supersymmetric boundary conditions, for the $\mathbf{CP}^{m+n-1|n}$ sigma models and for the continuum models constructed here (with weak coupling in the ultraviolet) are in both cases the same as for $U_q(\mathrm{sl}_2)$ continuum models, which become the O(3) sigma model for the $q = 1$ ($m = 2$) case. This is a consequence of the supersymmetry, which leads to a cancellation of corresponding bosonic and fermionic states in the familiar way. In general, it does not imply that all energy levels of theories that have equal partition functions are the same (if 'partition function' means $\mathrm{STr}\, \mathrm{e}^{-\beta H}$ [29] for supersymmetric formulations, and the q-deformed trace for the $U_q(\mathrm{sl}_2)$ formulation), since some may occur in multiplets of vanishing super- (or q-) dimension that do not contribute to the supertrace STr (respectively, q-deformed trace). However, for the spin chains constructed here within the TL algebra, all the energy levels for different models do coincide, at least in the $m = 2$ case, since the spin-1/2 (XXZ or TL) representation of the TL algebra is faithful, as is the supersymmetric one (for $n > 0$), and this result remains true in the continuum limit. For the spin chains of section 2.3, which lead to the $\mathbf{CP}^{m+n-1|n}$ models in the continuum limit, it is not clear to us if all the levels are in fact the same as those in the other two models for each m.

While the continuum models constructed here are integrable, the standard arguments for integrability break down in the $\mathbf{CP}^{m+n-1|n}$ sigma model with $n > 0$ for $m \neq 2$ [48]. Thus it seems unlikely that the $m = 1$ case of these flows or spin chains agrees with the $\mathbf{CP}^{m+n-1|n}$ sigma models. In other words they are different theories, even though the partition functions coincide. For $m = 2$, one suspects that the general $n > 0$ model is integrable, because of the usual arguments that the physics is the same for all n, and that its spectrum, S-matrices, etc, are related to those of the $q = 1$ SU(2) model. Thus some kind of limited equivalence of these $\mathrm{sl}(m+n|n)$-invariant theories is not ruled out. This leaves us uncertain how to describe the relation of the different theories, and whether all are integrable to the same extent. For the fixed point at $\theta = \pi$, and as a consequence also for the flow out of it, we believe that the theories coincide, as the spin chain models for $\mathcal{S} = 1/2$ do, so the results of the earlier sections are not affected.

We may note in passing that there are also natural constructions of boundary S-matrices with a spin-$\mathcal{S}$ on the boundary, for any $\mathcal{S}$, in the models introduced in this section. The boundary S-matrices would be found by fusion, as described for the Kondo problem in [35]. These would naturally describe different boundary conditions on the sigma model, or the spin chains depending on the $\mathcal{S}$ at each site of the chain and the degree of staggering of the nearest neighbour couplings. However, except when the boundary $\mathcal{S}$ is 0 or $1/2$, these representations of $\mathrm{sl}(m+n|n)$ are not those expected in the $\mathbf{CP}^{m+n-1|n}$ sigma models, as discussed in the previous section. The representations in the two cases are the same as those for the respective spin chains.

These S-matrices now exhaust the $U_q(\mathrm{sl}_2)$-invariant S-matrices known to us that could be used in different representations of the TL algebra. On the other hand, there are representations of the SO(3) BMW algebra that do not arise from representations of the TL algebra. A case in point is where the strands in figure 5 depict the vector representation of $\mathrm{OSp}(3+2n|2n)$, or more generally of $U_q(\mathrm{osp}(3+2n|2n))$. This is the basic construction of the SO(3) BMW algebra, extended to orthosymplectic supersymmetry [49]. For example, for $q = 1$, it is related to $\mathrm{SU}(2)_1/4$-state–Potts. Similarly, we could use representations of the TL algebra with $U_q(\mathrm{sl}(2+n|n))$ supersymmetry. In both cases, the properties will be related to the $n = 0$ cases as in earlier examples in this paper.

7. Conclusion

We have found S-matrices for the supersymmetric sigma models with $m = 1, 2$ that correspond to the flow from the fixed point at $\theta = \pi$ to the massive theory at $\theta = 0$. These results have potential applications to the spin quantum Hall transition in disordered fermion systems.

One can use the S-matrix to compute various physical quantities. For example, universal amplitude rations for the Potts models were computed in [50]. The thermodynamic Bethe ansatz computation for the Potts field theory was done in [51], but unfortunately the results are valid only at integer values $p \geqslant 3$, and so cannot be applied to the percolation limit $Q \to 1$. Perhaps our results will be useful in extending these computations further.

One interesting open question is to understand the relation between the two types of sigma models discussed in section 6. These models seem to be thermodynamically equivalent: does that mean that they have the same S-matrix?

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Appendix. SU(m)-invariant S-matrices for $m > 2$?

In this appendix we consider a question that arises naturally from the discussion in section 4: can we find exact S-matrices for the massive kinks in the $m > 2$ cases, assuming they always alternate kink and anti-kink? These might resemble exact S-matrices for the $\mathbf{CP}^{m-1}$ sigma model (if we set $n = 0$), however, similar questions to those raised in section 6 apply here as well. The $\mathbf{CP}^{m-1}$ sigma model is usually thought (though not proved) not to be integrable at $\theta = 0$ if $m > 2$; for $\theta = \pi$, we know of no conclusive arguments either way. By continuing the earlier results to $m > 2$, we find seemingly-sensible S-matrices, but they have a peculiar periodicity in the rapidity that rules them out from describing the sigma models. (Conceivably, at $\theta = \pi$, the assumption that the kinks and anti-kinks alternate is incorrect when the transition at $\theta = \pi$ is first-order.) Nonetheless, some of these S-matrices contradict a ‘proof’ of [52] that there is no S-matrix for particles in the adjoint of SU(m) for general m.

Since $m > 2$, we can set $n = 0$ in this appendix (the results generalize easily to $n > 0$). We first study the case where particles are in the fundamental representation of SU(m), and the particles alternate in space between the fundamental representation V (denoted here as m), and the dual V^* (denoted $\overline{m}$). The two-particle S-matrix in this model therefore is between a particle in the m representation and one in the $\overline{m}$, which we denote $S_{m\overline{m}} = S_{\overline{m}m}$, as it is independent of which is which. Because the particles must always alternate between m and $\overline{m}$, there is no S_{mm} or $S_{\overline{m}\overline{m}}$ in this model. In order to preserve the SU(m) symmetry, we must have

$$S_{m\overline{m}}(u) \propto f^{(0)}(u)\mathcal{P}^{(0)} + f^{(a)}(u)\mathcal{P}^{(a)},$$

where $\mathcal{P}^{(0)}$ and $\mathcal{P}^{(a)}$ project the tensor product of the m and $\overline{m}$ onto the singlet and adjoint representations. In section 3 we saw that these operators are

$$\mathcal{P}^{(0)} = \frac{e}{m} \qquad \mathcal{P}^{(a)} = 1 - \frac{e}{m}.$$

Using this, we can rewrite the S-matrix as

$$S_{m\overline{m}}(u) = Z(u)(I + F(u)e). \tag{28}$$

The function $F(u)$ is found by demanding $S_{m\overline{m}}$ that satisfy the YB equation, which here is

$$\begin{aligned}(I + F(u)e_i)(I + F(u+u')e_{i+1})(I + F(u')e_i)\\ = (I + F(u')e_{i+1})(I + F(u+u')e_i)(I + F(u')e_{i+1}).\end{aligned}$$

Using the TL relations (1), one finds that

$$F(u) + F(u') + NF(u)F(u') + F(u)F(u+u')F(u') - F(u+u') = 0.$$

The only non-trivial solution of this functional equation is

$$F(u) = -\frac{\sinh[Au]}{\sinh[\pi\widetilde{\gamma} + Au]} \tag{29}$$

where

$$N = 2\cosh(\pi\widetilde{\gamma})$$

and the parameter A is not constrained by the YB equation. Note that we have defined $\widetilde{\gamma}$ so that it is related to our earlier parameter γ by

$$\pi\widetilde{\gamma} = i\left(\gamma - \frac{\pi}{2}\right). \tag{30}$$

Earlier, we were interested in $m \leqslant 2$, where γ was real. Now we are interested in $m > 2$, where $\widetilde{\gamma}$ is real. This is a familiar solution of the YB equation when γ is real: it is that associated with the sine-Gordon S-matrix and the O(m) model S-matrix for $m < 2$ (as explained carefully in [7, 8]). In lattice-model language, when γ is real this is the solution of the YB equation associated with the six-vertex model and the RSOS models. Thus what we have done here is to argue that $S_{m\overline{m}}$ is essentially the continuation of a very familiar S-matrix to $m > 2$.

To complete the determination of the S-matrix, we need to find the function $Z(\theta)$ and the constant A. Unitarity requires that $S_{m\overline{m}}(u)S_{m\overline{m}}(-u) = I$, so

$$Z(u)Z(-u) = 1.$$

Crossing symmetry requires that S-matrix elements are related to those 'rotated' by 90°, namely

$$S(a\overline{b} \to c\overline{d})(u) = S(\overline{b}d \to \overline{a}c)(\mathrm{i}\pi - u).$$

The simplest way of satisfying this relation is to have

$$A = -\mathrm{i}\pi\widetilde{\gamma} + j,$$
$$Z(u) = Z(\mathrm{i}\pi - u)\frac{\sin[\widetilde{\gamma}(\mathrm{i}\pi - u)]}{\sin[\widetilde{\gamma}\theta]}$$

where j is some integer. We set $j = 0$, because a non-zero j will mean that the S-matrix is no longer unitary. The simplest $Z(\theta)$ obeying these relations is then

$$Z(u) = \prod_{p=1}^{\infty} \frac{\sin[\widetilde{\gamma}((2p-1)\mathrm{i}\pi - u)]\sin[\widetilde{\gamma}(2p\mathrm{i}\pi + u)]}{\sin[\widetilde{\gamma}((2p-1)\mathrm{i}\pi + u)]\sin[\widetilde{\gamma}(2p\mathrm{i}\pi - u)]}. \tag{31}$$

Note that A is purely imaginary, so that the S-matrix has periodicity under $u \to u + 2\pi/\widetilde{\gamma}$. Most S-matrices are periodic under *imaginary* shifts of rapidity, not real ones. The function $Z(u)$ is an elliptic function: in addition to the periodicity $Z(u) = Z(u + 2\pi/\widetilde{\gamma})$, it also has a quasi-periodicity under imaginary shifts in u, namely

$$Z(u + 2\mathrm{i}\pi) = Z(u)\frac{\sin^2[\widetilde{\gamma}(\mathrm{i}\pi + u)]}{\sin[\widetilde{\gamma}u]\sin[\widetilde{\gamma}(u + 2\mathrm{i}\pi)]}.$$

One can presumably use this to rewrite $Z(u)$ in terms of the usual elliptic θ functions.

An S-matrix periodic in real rapidity is unusual, to say the least. Such S-matrices have been discussed several times in the literature. In [53], an S-matrix based on the R-matrix for the lattice eight-vertex model was discussed. In [54], an elliptic generalization of the sinh-Gordon S-matrix was discussed, and resulting form factors were computed. In both these earlier cases, it was not clear which (if any) quantum field theories possessed such S-matrices. Here, we were motivated by the $\mathbf{CP}^{m-1}$ sigma model at $\theta = \pi$ to study a theory where particles alternate between the m and $\overline{m}$ representations of SU(m). However, this S-matrix does not seem to describe the sigma model. If the massive kinks described here are the excitations at $\theta = \pi$, then the ultraviolet limit of the S-matrix would describe the weakly-coupled sigma model at $\theta = \pi$. This is evidently not the case. Precisely, one can compute the energy in the presence of a background field from the S-matrix, and from the weak-coupling expansion sigma model of the sigma model. These two do not agree; the weak-coupling perturbation expansion shows no evidence of periodicity in the S-matrix.

It is expected that the $\mathbf{CP}^{m-1}$ model at $\theta = 0$ exhibits confinement. This has been established for $m = 2$ and large m, and is believed to hold for all $m > 0$ [30]. Confinement means that even though the fields are in the fundamental representations, no particles in the fundamental representations appear in the spectrum. Instead, the particles are believed to be in the adjoint representation of SU(m). This has been proved for $m = 2$, where it has long been known that the particles are in the three-dimensional representation of SU(2) [1].

One can use our results to construct an S-matrix for particles in the adjoint of SU(m). There is a claim in the literature that the only such S-matrix must be O($m^2 - 1$) invariant; we will show here that this is not so. As noted in section 4.2, with the appropriate representation of the TL algebra, the Potts S-matrix describes scattering of particles in the adjoint of SU(m) for $m \leqslant 2$. We have just shown that the O(m) model S-matrix for $m < 2$ can be 'continued' to $m > 2$. The same goes for the Potts S-matrix described in section 4.2, or for the spin-1 $U_q(\mathrm{sl}_2)$S-matrix described in section 6. For the former, we consider an S-matrix of the same form as (17), namely

$$S_{\mathrm{adj}}(u) = Z_{\mathrm{adj}}(u)(g(u)I + h(u)E + X)$$

where E and X obey the algebraic relations (18), (19), with as always $Q = m^2$. Then, as discussed above, S_{adj} obeys the YB equation if the functions $g(u)$ and $h(u)$ satisfy (20), where

$m = 2\sin\gamma$. However, this S-matrix is very different from the Potts S-matrix because when $m > 2$, γ is no longer real; it must be of the form (30) with $\widetilde{\gamma}$ real. Now if we take E and X to be defined in terms of the TL generator e as in (25), and we take the representation of the TL algebra as in the SU(m) quantum spin chains, then S_{adj} indeed describes the scattering of particles in the adjoint representation of SU(m). Such an S-matrix is not O($m^2 - 1$) invariant.

To complete this S-matrix, we need to find the function $Z_{\mathrm{adj}}(u)$. By using unitarity and crossing, we find that

$$g(u) = \frac{\sin[\widetilde{\gamma}(2\mathrm{i}\pi - 3u)]}{\sin[3\widetilde{\gamma}u]}$$
$$Z_{\mathrm{adj}}(u) = Z_{\mathrm{adj}}(\mathrm{i}\pi - u)$$
$$Z_{\mathrm{adj}}(u)Z_{\mathrm{adj}}(-u)g(u)g(-u) = 1.$$

The simplest solution for Z_{adj} is therefore

$$Z_{\mathrm{adj}}(u) = \frac{1}{g(-u)}\prod_{p=1}^{\infty}$$
$$\times \frac{\sin[\widetilde{\gamma}((6p-3)\mathrm{i}\pi - u)]\sin[\widetilde{\gamma}(6p\mathrm{i}\pi + u)]\sin[\widetilde{\gamma}((6p+2)\mathrm{i}\pi - u)]\sin[\widetilde{\gamma}((6p-1)\mathrm{i}\pi + u)]}{\sin[\widetilde{\gamma}((6p-3)\mathrm{i}\pi + u)]\sin[\widetilde{\gamma}(6p\mathrm{i}\pi - u)]\sin[\widetilde{\gamma}((6p+2)\mathrm{i}\pi + u)]\sin[\widetilde{\gamma}((6p-1)\mathrm{i}\pi - u)]}.$$

Like the S-matrix for particles alternating between m and $\overline{m}$ representations, this is periodic under real shifts in rapidity and quasi-periodic under imaginary shifts.

The other SU(m)S-matrices for particles in the adjoint that are the continuation of those in section 6 to $m > 2$, also exhibit the same peculiar periodicity. Our conclusion is therefore that these S-matrices do not describe either the $\theta = \pi$ or $\theta = 0$ $\mathbf{CP}^{m-1}$ models. They join the S-matrices of [53, 54] as interesting S-matrices with no known physical interpretation.

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